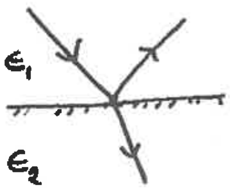
PLANE WAVES

Gaussian units



We will be interested in plane electromagnetic waves propagating in linear media ($\underline{D} = \epsilon \underline{E}$, $\underline{B} = \mu \underline{H}$), and in reflection and refraction at plane boundaries where two such media meet.

In the medium ϵ (and perhaps also μ) differ from 1 due to interaction with molecules. These can be thought of as harmonic oscillators, with various resonance frequencies etc. This means that ϵ can depend on the frequency of the wave. For this reason we begin with a Fourier transformation, and treat each frequency separately.

Thus

$$\tilde{\underline{E}}(\underline{x}, \omega) = \int_{-\infty}^{\infty} \underline{E}(\underline{x}, t)\, e^{i\omega t}\, dt$$

$$\underline{E}(\underline{x}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{\underline{E}}(\underline{x}, \omega)\, e^{-i\omega t}\, dt$$

and similarly for $\underline{B}$. We assume ϵ, μ to be independent of $\underline{x}$, drop the tildes, and obtain Maxwell's eqs.

$$\underline{\nabla} \cdot \underline{E} = 0 \qquad \underline{\nabla} \times \underline{E} - i \frac{\omega}{c} \underline{B} = 0$$

$$\underline{\nabla} \cdot \underline{B} = 0 \qquad \underline{\nabla} \times \underline{B} + i \frac{\omega \mu \epsilon}{c} \underline{E} = 0$$

$\Rightarrow$

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{E}) = i \frac{\omega}{c} \underline{\nabla} \times \underline{B} = \frac{\omega^2 \mu \epsilon}{c^2} \underline{E}$$

So

$$\nabla^2 \underline{E}(\underline{x}, \omega) + \frac{\omega^2 \mu \epsilon}{c^2} \underline{E}(\underline{x}, \omega) = 0$$

and similarly for $\underline{B} = \underline{B}(\underline{x}, \omega)$

Let $\underline{\epsilon}_1$, $\underline{\epsilon}_2$, $\underline{n}$ be unit vectors in space.

We solve the wave equations by

$$\underline{E} = \underline{\epsilon}_1 A \cos(\underline{k}\cdot\underline{x} - \omega t + \text{phase})$$

$$\underline{B} = \underline{\epsilon}_2 B \cos(\underline{k}\cdot\underline{x} - \omega t + \text{phase}')$$

where $\underline{k} = k\,\underline{n}$ and

$$k^2 = \frac{\omega^2 \mu\epsilon}{c^2}$$

To deal with the phases in the solution, one can set

$$\underline{E} = \text{Re}\left[\underline{\epsilon}_1 \alpha e^{i(\underline{k}\cdot\underline{x} - \omega t)}\right]$$

where α is complex. Because the equations are linear in $\underline{E}$ (and $\underline{B}$), eg

$$\nabla \times (\text{Re}[\underline{E}]) = \text{Re}[\underline{\nabla} \times \underline{E}]$$

We can do all the calculations using complex fields, if we find it convenient.

Maxwell's eqs. contain some more details:

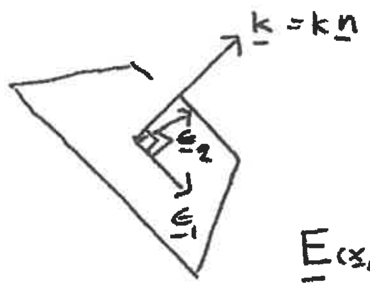
$$\underline{\nabla}\cdot\underline{E} = \underline{\nabla}\cdot\underline{B} = 0 \quad \Rightarrow \quad \underline{n}\cdot\underline{\epsilon}_1 = \underline{n}\cdot\underline{\epsilon}_2 = 0$$

$$\underline{B} = -\frac{ic}{\omega}\underline{\nabla}\times\underline{E} \quad \Rightarrow$$

$$\Rightarrow B\underline{\epsilon}_2 = \frac{ck}{\omega} A\, \underline{n}\times\underline{\epsilon}_1 \overset{!}{=} \sqrt{\mu\epsilon}\, A\, \underline{n}\times\underline{\epsilon}_1$$

$$\Rightarrow \underline{\epsilon}_1\cdot\underline{\epsilon}_2 = 0$$

So we have an ON-basis associated to the solution



We have one solution,

$$\underline{E}(\underline{x},\omega) = A\underline{\epsilon}_1 \cos(\underline{k}\cdot\underline{x} - \omega t + \text{phase})$$

$$\underline{B}(\underline{x},\omega) = \sqrt{\mu\epsilon}\, A\underline{\epsilon}_2 \cos(\underline{k}\cdot\underline{x} - \omega t + \text{phase})$$

Notice that the electric and magnetic field are in phase with each other.

But now we can write down two linearly independent solutions

I adjusted "t" to get rid of the phase here

$$\underline{E}_1 = E_1 \underline{\epsilon}_1 \cos(\underline{k}\cdot\underline{x} - \omega t)$$

$$\underline{E}_2 = E_2 \underline{\epsilon}_2 \cos(\underline{k}\cdot\underline{x} - \omega t + \text{phase})$$

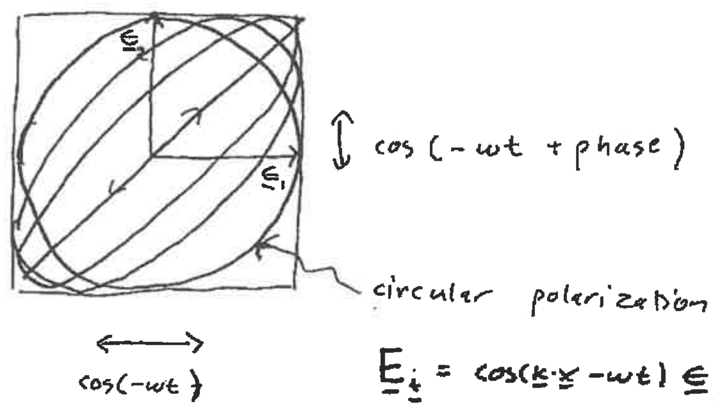
(The magnetic fields can be written down easily, so I don't write them.)

These solutions are said to be linearly polarized. The general solution is a linear combination of the two.

And unless the phase is set to zero, the direction of the electric field will rotate in the $(\underline{\epsilon}_1, \underline{\epsilon}_2)$-plane.

To see this, recall the Lissajous figures. Choose $E_1 = E_2$ for simplicity.

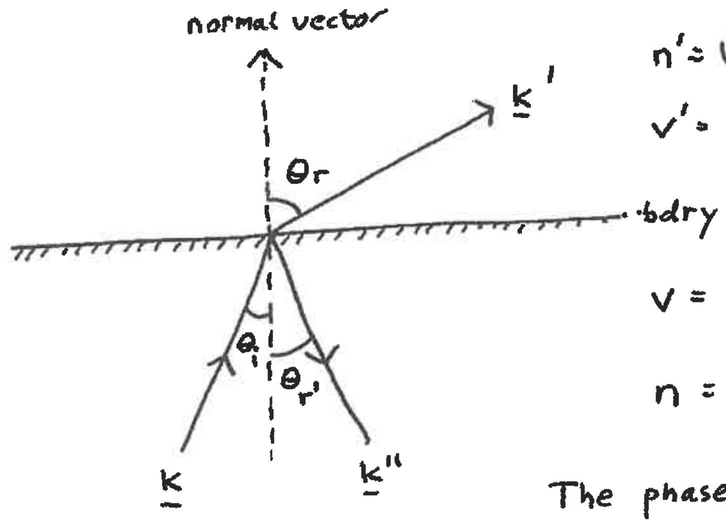
Then, as you vary the phase, you will get straight lines (linear polarization), ellipses (elliptic polarization), or circles (circular polarization):



$$\underline{E}_{\pm} = \cos(\underline{k}\cdot\underline{x} - \omega t)\,\underline{\epsilon}_1 + \pm\sin(\underline{k}\cdot\underline{x} - \omega t)\,\underline{\epsilon}_2$$

NB: If you want to go deeper into this, the complex notation Jackson uses wins hands down.

Now let a plane wave hit the (plane) boundary between two media.



$n' = \sqrt{\mu' \epsilon'}$

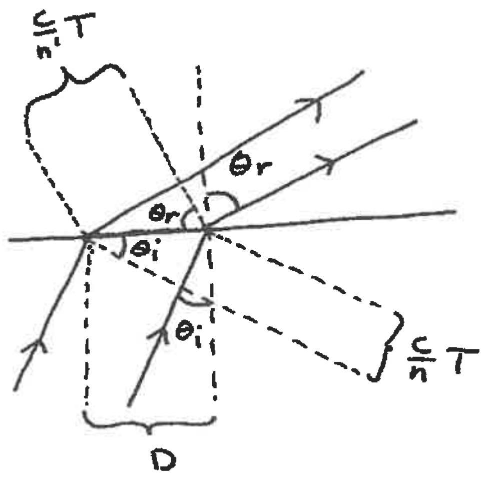
$v' = \frac{c}{n'}$

$v = \frac{c}{n}$

$n = \sqrt{\mu \epsilon}$

The phase velocities v, v' differ between the two media.

The relation between θ_i and θ_r are easily found by considering planes of equal phase:



$$\frac{c}{n} T = \sin\theta_i \, D$$

$$\frac{c}{n'} T = \sin\theta_r \, D$$

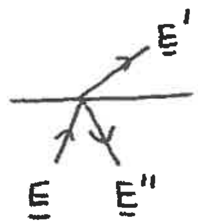
$$\Rightarrow \frac{\sin\theta_i}{\sin\theta_r} = \frac{n'}{n}$$

Similarly $\theta_i = \theta_r'$

We want the amplitude and the polarization of the refracted (reflected) wave

This is determined by the fact that the normal components of $\underline{D}$ and $\underline{B}$, and the tangential components of $\underline{E}$ and $\underline{H}$, are continuous.

It is therefore convenient to consider two linear polarizations, $\underline{E}$ tangential to the boundary (= orthogonal to the "plane of incidence), and $\underline{E}$ tangential to the plane of incidence.



The general solution is a linear combination of those.

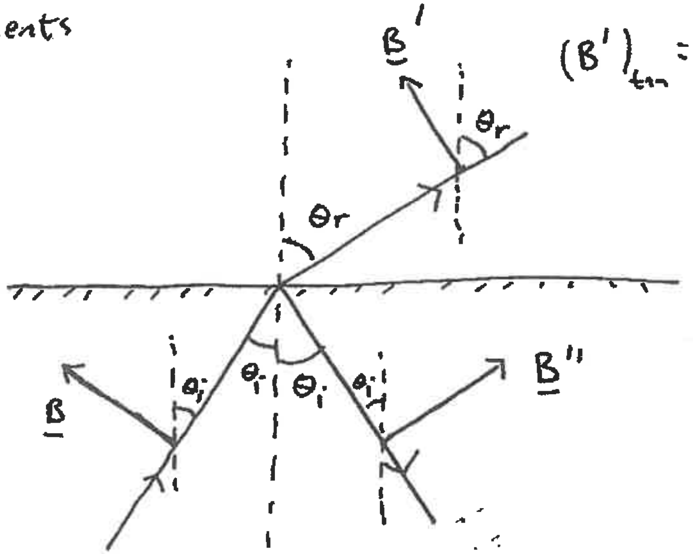
All the amplitudes are determined by the amplitude of the electric field; at the boundary

$$\underline{E}_0 \,, \quad \underline{D}_0 = \epsilon \underline{E}_0 \,,$$

$$\underline{B}_0 = \sqrt{\mu\epsilon}\; \underline{n} \times \underline{E}_0 \,, \quad \underline{H}_0 = \sqrt{\frac{\epsilon}{\mu}}\; \underline{n} \times \underline{E}_0$$

as in $\underline{k} = k\underline{n}$

Let us consider the case when $\underline{E}$ is tangential to the boundary. $\underline{B}$ will have both normal and tangential components



$$(B')_{tan} = B' \cos\theta_r$$

$$(B)_{tan} = B\cos\theta_i \qquad (B'')_{tan} = -B\cos\theta_i$$

tangential $\underline{E}$:

$$E_0 + E_0'' = E_0'$$

tangential $\underline{H}$:

$$\sqrt{\frac{\epsilon}{\mu}}\,(E_0 - E_0'')\cos\theta_i = \sqrt{\frac{\epsilon'}{\mu'}}\, E_0' \cos\theta_r$$

normal $\underline{B}$:

$$\sqrt{\mu\epsilon}\,(E_0 + E_0'')\sin\theta_i = \sqrt{\mu'\epsilon'}\, E_0' \sin\theta_r$$

Given Snell's law, the third equation is the same as the first ($\sqrt{\mu\epsilon}\sin\theta_i = \sqrt{\mu'\epsilon'}\sin\theta_r$).

It is then a simple matter to solve for

$$\frac{E_0'}{E_0} = \frac{2\sqrt{\epsilon\mu}\cos\theta_i}{\sqrt{\epsilon\mu}\cos\theta_i + \frac{\mu}{\mu'}\sqrt{\epsilon'\mu'}\cos\theta_r}$$

$$\frac{E_0''}{E_0} = \frac{\sqrt{\epsilon\mu}\cos\theta_i - \frac{\mu}{\mu'}\sqrt{\epsilon'\mu'}\cos\theta_r}{\sqrt{\epsilon\mu}\cos\theta_i + \frac{\mu}{\mu'}\sqrt{\epsilon'\mu'}\cos\theta_r}$$

The other case ($\underline{E}$ in the plane of incidence, $\underline{B}$ tangential to the boundary) can be done similarly.

Note: $\sqrt{\epsilon\mu} = n$

$$n' \cos\theta_r = n' \sqrt{1-\sin^2\theta_r} = \sqrt{n'^2 - n^2 \sin^2\theta_i}$$

The results, in the two cases:

$\underline{E}$ tangential to boundary:

$$\frac{E_0'}{E_0} = \frac{2n\cos\theta_i}{n\cos\theta_i + \frac{\mu}{\mu'}\sqrt{n'^2-n^2\sin^2\theta_i}}$$

$$\frac{E_0''}{E_0} = \frac{n\cos\theta_i - \frac{\mu}{\mu'}\sqrt{n'^2-n^2\sin^2\theta_i}}{n\cos\theta_i + \frac{\mu}{\mu'}\sqrt{n'^2-n^2\sin^2\theta_i}}$$

$\underline{B}$ tangential to boundary:

$$\frac{E_0'}{E_0} = \frac{2nn'\cos\theta_i}{\frac{\mu}{\mu'}n'^2\cos\theta_i + n\sqrt{n'^2-n^2\sin^2\theta_i}}$$

$$\frac{E_0''}{E_0} = \frac{\frac{\mu}{\mu'}n'^2\cos\theta_i - n\sqrt{n'^2-n^2\sin^2\theta_i}}{\frac{\mu}{\mu'}n'^2\cos\theta_i + n\sqrt{n'^2-n^2\sin^2\theta_i}}$$

These formulas contain interesting physics.

Consider the second case, $\underline{B}$ tangential:

At optical frequencies $\mu \approx \mu'$!

The reflected wave will vanish if

$$n'^2\cos\theta_i = n\sqrt{n'^2-n^2\sin^2\theta_i}$$

$$\Leftrightarrow$$

$$\sin^2\theta_i = \frac{n'^2}{n'^2+n^2}$$

$$\Leftrightarrow$$

$$\tan\theta_i = \frac{n'}{n}$$

This is "Brewster's angle".

If unpolarized light hits the boundary at Brewster's angle, the reflected light will be linearly polarized with $\underline{E}$ (not $\underline{B}$!) tangential to the boundary.