A NEW NOTATION FOR QUANTUM MECHANICS

In mathematical theories the question of notation, while not of primary importence, is yet worthy of careful consideration, since a good notation can be of great help in the development of a theory, by making it easy to write down those quantities that are important, and difficult or impossible to write down those that are unimportant.

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A vector: $|\psi\rangle$

A dual vector: $\langle\varphi|$

A number: $\langle\varphi|\psi\rangle$

An operator: $|\psi\rangle\langle\varphi|$

A general operator, expressed using a basis:

$$A = \sum_{i,j} A^i{}_j \, |e_i\rangle\langle e^j|$$

Tensor products put a strain on this:

A product basis is

$$|e_i\rangle \otimes |f_j\rangle = |e_i\rangle|f_j\rangle$$

(For many qubits, you often see

$$|0010\rangle = |e_0\rangle \otimes |e_0\rangle \otimes |e_1\rangle \otimes |e_0\rangle \text{ etc.})$$

Dual vectors

$$\langle e^i| \otimes \langle f^j| = \langle e^i|\langle f^j|$$

I sometimes use an extra label, and can then write things like

$$|e_i\rangle_1 |e_j\rangle_2 \leftrightarrow {}_1\langle e^i| \, {}_2\langle e^j|$$

Note two different ways of writing the operator

$$\boxed{|e\rangle \otimes |f\rangle\, \langle e| \otimes \langle f| \overset{!}{=} |e\rangle\langle e| \otimes |f\rangle\langle f|}$$

This is a <u>local</u> operator, of the form .

$$A \otimes B$$

A general operator on a bipartite Hilbert space is not like that.

A general operator can be expanded, using a product basis, as

$$U = \sum_{i,j,\alpha,\beta} U^{i\alpha}{}_{j\beta}\, |e_i\rangle\langle e^j| \otimes |f_\alpha\rangle\langle f^\beta|$$

Traces :

$$\mathrm{Tr}\, A = \sum_i \langle e^i | A | e_i \rangle =$$

$$= \sum_{i,j,k} \langle e^i | A^j{}_k | e_j \rangle \langle e^k | e_i \rangle =$$

$$= \sum_{i,j,k} A^j{}_k \langle e^i | e_j \rangle \langle e^k | e_i \rangle =$$

$$= \sum_{i,j,k} A^j{}_k \, \delta^i{}_j \, \delta^k{}_i =$$

$$= \sum_i A^i{}_i \quad = \text{a } \underline{\text{number}},$$

containing some of the information in A

The <u>partial trace</u> of an operator acting on $\mathcal{H}_1 \otimes \mathcal{H}_2$ is an operator acting on one of the factors. For instance, tracing out the second factor

$$\mathrm{Tr}_2\, \mathcal{U} = \sum_\alpha \langle f^\alpha | \mathcal{U} | f_\alpha \rangle =$$

$$= \sum_{\substack{i,j,\\ \alpha,\beta,\gamma}} \langle f^\alpha | \mathcal{U}^{i\beta}{}_{j\gamma} |e_i\rangle\langle e^j| \otimes |f_\beta\rangle\langle f^\gamma | f_\alpha \rangle$$

$$= \sum_{i,j,\alpha,\beta,\gamma} \mathcal{U}^{i\beta}{}_{j\gamma} |e_i\rangle\langle e^j| \otimes \langle f^\alpha | f_\beta \rangle \langle f^\gamma | f_\alpha \rangle$$

$$= \sum_{i,j,\alpha} \mathcal{U}^{i\alpha}{}_{j\alpha} |e_i\rangle\langle e^j|$$