FK8024: Adv. GR

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Problem 6.1 The Kerr Spacetime

Statement: Calculate $\sqrt{g}$ for the Kerr metric in Boyer-Lindquist coordinates. Also, calculate, by hand, at least two non-zero Christoffel symbols. Conclusions?

Kerr metric in Boyer-Lindquist coordinates:
(line-element)

$$ds^2 = -\frac{\Delta\rho^2}{A}dt^2 + \frac{\rho^2 dr^2}{\Delta} + \rho^2 d\theta^2 + \frac{F\sin^2\theta}{\rho^2}(d\phi - \omega dt)^2$$

where:

$$\rho^2 \equiv r^2 + a^2\cos^2\theta$$

$$\Delta \equiv r^2 - 2mr + a^2$$

$$F \equiv (r^2+a^2)^2 - a^2\Delta\sin^2\theta$$

$$\omega \equiv \frac{2mar}{F}$$

$$A \equiv \frac{\Delta\rho^4 a}{a\rho^2 - F\omega + F\omega^2 a\sin^2\theta}$$

Noting the pairings of differential forms, it is possible to write the metric-tensor in matrix form:

$$\begin{pmatrix} \frac{F\omega^2\sin^2\theta}{\rho^2} - \frac{\Delta\rho^2}{A} & 0 & 0 & -\frac{F\omega\sin^2\theta}{\rho^2} \\ 0 & \frac{\rho^2}{\Delta} & 0 & 0 \\ 0 & 0 & \rho^2 & 0 \\ -\frac{F\omega\sin^2\theta}{\rho^2} & 0 & 0 & \frac{F\sin^2\theta}{\rho^2} \end{pmatrix}$$

In this form computation of the metric determinant is simplified. A straight forward calculation then yields:

$$\sqrt{g} = \sqrt{\det g} = \left[\left(\frac{F\omega^2 \sin^2\theta}{\rho^2} - \frac{\Delta \rho^2}{A} \right) \left(\frac{\rho^2}{\Delta} \right) \left(F \sin^2\theta \right) + \left(\frac{\omega F \sin^2\theta}{\rho^2} \right) \left(-\frac{\rho^2}{\Delta} \right) \left(\omega F \sin^2\theta \right) \right]^{\frac{1}{2}}$$

Simplified this is:

$$\sqrt{g} = \left[-\frac{\rho^4 F \sin^2\theta}{A} \right]^{\frac{1}{2}}$$

If we substitute in the A term this is:

$$\sqrt{g} = \left[-\sin^2\theta \left(\frac{F(a\rho^2 - F\omega + F\omega^2 a \sin^2\theta)}{a\Delta} \right) \right]^{\frac{1}{2}}$$

Substitute in F and ω and ρ^2

$$\sqrt{g} = \left[-\sin^2\theta \left(\frac{((r^2+a^2)^2 - a^2\Delta\sin^2\theta)\left(a(r^2+a^2\cos^2\theta) - 2mar + \frac{4m^2a^3r^2\sin^2\theta}{((r^2+a^2)^2 - a^2\Delta\sin^2\theta)}\right)}{a\Delta} \right) \right]^{\frac{1}{2}}$$

Common denom. and expand all terms:

$$\sqrt{g} = \left[-\sin^2\theta \left(\left(\frac{F}{\Delta} \right) \left(\frac{((r^2+a^2)^2 - a^2\Delta\sin^2\theta)\left[(r^2+a^2\cos^2\theta) - 2mr\right] + 4m^2a^2r^2\sin^2\theta}{F} \right) \right) \right]^{\frac{1}{2}}$$

This gives the following expression

$$\sqrt{g} = \Bigg[-\frac{\sin^2\theta}{\Delta} \Big(r^6 + 2a^2r^4 + a^4r^2 - a^2r^2\Delta\sin^2\theta + a^2r^4\cos^2\theta + 2a^4r^2\cos^2\theta \ldots$$

$$+ a^6\cos^2\theta - a^4\Delta\cos^2\theta\sin^2\theta - 2mr^5 - 4ma^2r^3 - 2ma^4r \ldots$$

$$+ 2ma^2r\Delta\sin^2\theta + 4m^2a^2r^2\sin^2\theta \Big) \Bigg]^{1/2}$$

Substitute $\Delta \equiv r^2 - 2mr + a^2$ and $\sin^2\theta = 1 - \cos^2\theta$: Full expansion

$$\sqrt{g} = \Bigg[-\frac{\sin^2\theta}{\Delta} \Big(r^6 - 2mr^5 + 2a^2r^4 + a^4r^2 - 4ma^2r^3 - 2ma^4r \ldots$$

$$+ a^2r^4\cos^2\theta + 2a^4r^2\cos^2\theta + a^6\cos^2\theta - a^2r^4 + 2ma^2r^3 \ldots$$

$$- a^4r^2 + a^2r^4\cos^2\theta - 2ma^2r^3\cos^2\theta + a^4r^2\cos^2\theta + 2ma^2r^3 \ldots$$

$$- 4m^2a^2r^2 + 2ma^4r - 2ma^2r^3\cos^2\theta + 4m^2a^2r^2\cos^2\theta \ldots$$

$$- 2ma^4r\cos^2\theta + 4m^2a^2r^2 - 4m^2a^2r^2\cos^2\theta - a^4r^2\cos^2\theta \ldots$$

$$+ 2ma^4r\cos^2\theta - a^6\cos^2\theta + a^4r^2\cos^4\theta - 2ma^4r\cos^4\theta \ldots$$

$$+ a^6\cos^4\theta \Big) \Bigg]^{1/2}$$

This ugly expression can be simplified if we eliminate and group like terms.

$$\sqrt{g} = \Bigg[-\frac{\sin^2\theta}{\Delta} \Big(r^6 - 2mr^5 + a^2r^4 + 2a^2r^4\cos^2\theta - 4ma^2r^3\cos^2\theta \ldots$$

$$+ 2a^4r^2\cos^2\theta + a^4r^2\cos^4\theta - 2ma^4r\cos^4\theta + a^6\cos^4\theta \Big) \Bigg]^{1/2}$$

continued...

Looking at the previous expression closely we see that $\Delta \equiv r^2 - 2mr + a^2$ can be factored out so then:

$$\sqrt{g} = \left[-\sin^2\theta \left(r^4 + 2a^2 r^2 \cos^2\theta + a^4 \cos^4\theta \right) \right]^{1/2}$$

replacing the right-term with $\rho^4 = r^4 + 2a^2 r^2 \cos^2\theta + a^4 \cos^4\theta$

we have the simplest form

$$\sqrt{g} = \left[-\sin^2\theta \, \rho^4 \right]^{1/2}$$

usually (in the action formulation of GR) we want $\sqrt{-g}$ and we see this is elegantly:

$$\sqrt{-g} = \rho^2 \sin\theta \quad \text{where } \rho^2 \equiv r^2 + a^2\cos^2\theta$$

Now, I move on to calculation of two non-zero Christoffel symbols. Recall:

$$\Gamma^{\mu}_{\alpha\beta} = \frac{g^{\mu\nu}}{2} \left[g_{\alpha\nu,\beta} + g_{\beta\nu,\alpha} - g_{\alpha\beta,\nu} \right]$$

Noting that the inverse metric-tensor is needed compute this with the result of $\det(g) = -\rho^4 \sin^2\theta$

$g^{\mu\nu}$ is then

$$\begin{pmatrix} -\frac{A}{\Delta\rho^2} & 0 & 0 & -\frac{A\omega}{\Delta\rho^2} \\ 0 & \frac{\Delta}{\rho^2} & 0 & 0 \\ 0 & 0 & \frac{1}{\rho^2} & 0 \\ -\frac{A\omega}{\Delta\rho^2} & 0 & 0 & \frac{\Delta\rho^6 - FA\rho^2\omega^2\sin^2\theta}{F\Delta\rho^4\sin^2\theta} \end{pmatrix}$$

For simplicity in choosing which Christoffel symbols to calculate we know all terms are independent of t So we want to select Γ^{μ}_{tt} to limit the number terms to compute but this also implies $\Gamma^{t}_{tt} \equiv \Gamma^{0}_{00} = 0$

So I will compute $\Gamma^{r}_{tt} \equiv \Gamma^{1}_{00}$ & $\Gamma^{\theta}_{tt} \equiv \Gamma^{2}_{00}$

$$\Gamma^{1}_{00} = \frac{g^{1}}{2}\left[g_{01,0} + g_{01,0} - g_{00,1}\right] + g^{10}$$

As I mentioned, no explicit time dependance. so only need to compute $g_{00,1}$

$$g_{00,1} = \frac{\partial}{\partial r}\left(\frac{F\omega^2\sin^2\theta}{\rho^2} - \frac{\Delta\rho^2}{A}\right)$$

$$= \frac{\rho^2\sin^2\theta\,\partial_r(F\omega^2) - F\omega^2\sin^2\theta\,\partial_r(\rho^2)}{\rho^4} - \frac{A\partial_r(\Delta\rho^2) - \Delta\rho^2\partial_r(A)}{A^2}$$

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Compute the derivatives $\partial_r(F\omega^2), \partial_r(\rho^2), \partial_r(\Delta\rho^2), \partial_r A$

$$\partial_r(F\omega^2) = \partial_r\left(\frac{4m^2a^2r^2}{F}\right) = 4m^2a^2\,\partial_r\left(\frac{r^2}{F}\right)$$

$$= 4m^2a^2\left(\frac{2rF - r^2\partial_r F}{F^2}\right) = \frac{8m^2a^2rF - r^2\partial_r((r^2+a^2)^2 - a^2\Delta\sin^2\theta)}{F^2}$$

$$= \frac{8m^2a^2rF - r^2(4r^3 + (4-2\sin^2\theta)a^2r + 2ma^2\sin^2\theta)}{F^2}$$

$$= \frac{8m^2a^2rF - 4r^5 - (4-2\sin^2\theta)a^2r^3 - 2ma^2r^2\sin^2\theta}{F^2} \quad \square$$

$$\partial_r(\rho^2) = 2\rho\,\partial_r(r^2 + a^2\cos^2\theta)$$

$$= 2(r^2+a^2\cos^2\theta)^{1/2}(2r)$$

$$= 4r\sqrt{r^2+a^2\cos^2\theta} \quad \square$$

$$\partial_r(\Delta\rho^2) = \rho^2\partial_r\Delta + \Delta(4r\sqrt{r^2+a^2\cos^2\theta})$$

$$= (r^2+a^2\cos^2\theta)\left[\partial_r(r^2-2mr+a^2)\right] + (r^2-2mr+a^2)(4r\sqrt{r^2+a^2\cos^2\theta})$$

$$= (2r-2m)(r^2+a^2\cos^2\theta) + (4r\sqrt{r^2+a^2\cos^2\theta})(r^2-2mr+a^2)$$

$$= 2r^3 + 2ra^2\cos^2\theta - 2mr^2 - 2ma^2\cos^2\theta + (4r^3 - 8mr^2 + 4a^2r)\sqrt{r^2+a^2\cos^2\theta} \quad \square$$

$$\partial_r(A) = \partial_r\left(\frac{a\Delta\rho^4}{a\rho^2 - F\omega + F\omega^2a\sin^2\theta}\right)$$

$$= \frac{(a\rho^2 - F\omega + F\omega^2a\sin^2\theta)\,a\,\partial_r(\Delta\rho^4) - a\Delta\rho^4\,\partial_r(a\rho^2 - F\omega + F\omega^2a\sin^2\theta)}{(a\rho^2 - F\omega + F\omega^2a\sin^2\theta)^2}$$

$$= \frac{(a\rho^2 - F\omega + F\omega^2a\sin^2\theta)\,a\left[(\Delta 4\rho^3)2r + \rho^4(2r-2m)\right] - a\Delta\rho^4(2a\rho(2r) - 2ma + a\sin^2\theta)(F\omega^2}{(a\rho^2 - F\omega + F\omega^2a\sin^2\theta)^2}$$

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7/10

Continuing computation of $\partial_r A$

$$\partial_r A = \left(\frac{1}{(a\rho^2 - F\omega + F\omega^2 \sin^2\theta)^2}\right)\Big[8ar\rho^3\Delta(a\rho^2 - F\omega + F\omega^2 a\sin^2\theta) \dots$$

$$\dots + 2a\rho^4 r(a\rho^2 - F\omega + F\omega^2 a\sin^2\theta) - 2ma\rho^4(a\rho^2 - F\omega + F\omega^2\sin^2\theta) -$$

$$- 4a^2\rho^5 r\Delta + 2ma^2\rho^4\Delta - a^2\Delta\rho^4\sin^2\theta\left(\frac{8m^2a^4rF - 4r^5 - (4-2\sin^2\theta)a^2r^3 - 2ma^2r^2\sin^2\theta}{F^2}\right)\Big]$$

Then, putting all these terms together

$$g_{00,1} = \Bigg(-\frac{4a^2m^2r^2\sin^2\theta(4r(r^2+a^2) - a^2(2r-2m)\sin^2\theta)}{\rho^2((a^2+r^2)^2 - a^2(r^2-2mr+a^2)\sin^2\theta)^2} + \dots$$

$$\frac{8m^2a^2r\sin^2\theta}{\rho^2((a^2+r^2)^2 - a^2(r^2-2mr+a^2)\sin^2\theta)} + \frac{2m}{\rho^4}(\rho^2 - 2r^2) + \frac{4a^2m^2r^2\sin^2\theta(4r(r^2+a^2) - a^2(2r-2m)\sin^2\theta)}{\rho^2((a^2+r^2)^2 - a^2(r^2-2mr+a^2)\sin^2\theta)^2}$$

$$- \frac{8m^2a^2r\sin^2\theta}{\rho^2((a^2+r^2)^2 - a^2(r^2-2mr+a^2)\sin^2\theta)}\Bigg)$$

which drastically simplifies to

$$g_{00,1} = \frac{2m}{\rho^2}\left(1 - \frac{2r^2}{\rho^2}\right)$$

Then the full Christoffel symbol of the second kind is:

$$\boxed{\Gamma^1_{00} = \frac{m\Delta}{\rho^6}(2r^2 - \rho^2)} \quad \square$$

Now compute the second Christoffel symbol.

$$\Gamma^{2}_{00} = \frac{g^{22}}{2}\left[g_{02,0} + g_{02,0} - g_{00,2}\right]$$

Again, no explicit time dependence so

$$\Gamma^{2}_{00} = -\frac{g^{22}}{2} g_{00,2}$$

From the inverse metric $g^{22} = \frac{1}{\rho^2}$

So calculate $g_{00,2}$ i.e. $g_{tt,\theta}$

$$g_{00,2} = \left(\frac{\partial}{\partial\theta}\left(\frac{F\omega^2 \sin^2\theta}{\rho^2} - \frac{\Delta\rho^2}{A}\right)\right)$$

$$= \frac{\rho^2\left(F\omega^2(2\sin\theta\cos\theta) + \sin^2\theta\,\partial_\theta(F\omega^2)\right) - F\omega^2\sin^2\theta\, 2\rho\,\partial_\theta\rho}{\rho^4} \ldots$$

$$- \frac{A\left(2\rho\Delta\,\partial_\theta\rho + \rho^2\partial_\theta\Delta\right) - \Delta\rho^2\partial_\theta A}{A^2}$$

So again computing derivatives $\partial_\theta(F\omega^2), \partial_\theta\rho, \partial_\theta\Delta, \partial_\theta A$

$$\partial_\theta(F\omega^2) = -\frac{4m^2a^2r^2}{F^2}\partial_\theta F = -\frac{4m^2a^2r^2}{F^2}\left(-a^2\Delta\, 2\sin\theta\cos\theta\right)$$

$$\partial_\theta\rho = \partial_\theta(r^2 + a^2\cos^2\theta)^{\frac{1}{2}} = \frac{1}{2}(r^2 + a^2\cos^2\theta)^{-\frac{1}{2}} 2a^2\cos\theta(-\sin\theta)$$

$$\partial_\theta\Delta = 0$$

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$$\partial_\theta A = \frac{(a\rho^2 - F\omega + F\omega^2 a\sin^2\theta)\, a\Delta 4\rho^3 \partial_\theta \rho - a\Delta\rho^4 \partial_\theta(a\rho^2 - F\omega + F\omega^2 a\sin^2\theta)}{(a\rho^2 - F\omega + F\omega^2 a\sin^2\theta)^2}$$

$$= \Big[-4a\Delta\rho^3 (a^2\cos\theta\sin\theta)(r^2+a^2\cos^2\theta)^{\frac{1}{2}}(a\rho^2 - F\omega + F\omega^2 a\sin^2\theta) \ldots$$

$$-a\Delta\rho^4\Big(2a^3\rho\cos\theta\sin\theta(r^2+a^2\cos^2\theta)^{\frac{1}{2}} + \left(\frac{8m^2a^3r^2}{F}\right)\sin\theta\cos\theta - \frac{8m^2a^3r\sin^2\theta}{F^2} \ldots$$

$$\times\left(-a^2\Delta 2\sin\theta\cos\theta\right)\Big)\Big]\left(\frac{1}{(a\rho^2 - F\omega + F\omega^2 a\sin^2\theta)^2}\right)$$

Collect terms in $g_{00,2}$

$$g_{00,2} = \left[2\rho^2 F\omega^2\sin\theta\cos\theta + \rho^2\sin^2\theta\left(\frac{8m^2a^4r^2\Delta\sin\theta\cos\theta}{F^2}\right)\right]\frac{1}{\rho^4} \ldots$$

$$+\frac{1}{F\rho^3}\left[8m^2a^4r^2\sin^3\theta\cos\theta(r^2+a^2\cos^2\theta)^{\frac{1}{2}}\right] \ldots$$

$$+\left(\frac{1}{A}\right)\left(2\rho\Delta a^2\cos\theta\sin\theta(r^2+a^2\cos^2\theta)^{\frac{1}{2}}\right) + \left(\frac{\Delta\rho^2}{A^2}\right)\left(\frac{1}{(a\rho^2 - F\omega + F\omega^2 a\sin^2\theta)^2}\right)\Big[\ldots$$

$$\Big[-4a^3\Delta\rho^3\cos\theta\sin\theta\,(r^2+a^2\cos^2\theta)^{\frac{1}{2}}(a\rho^2 - F\omega + F\omega^2 a\sin^2\theta) \ldots$$

$$-a\Delta\rho^4\Big(-2a^3\rho\cos\theta\sin\theta(r^2+a^2\cos^2\theta)^{\frac{1}{2}} + \left(\frac{8m^2a^3r^2}{F}\right)\sin\theta\cos\theta - \frac{8m^2a^3r\sin^2\theta}{F^2}$$

$$\left(-2a^2\Delta\sin\theta\cos\theta\right)\Big)\Big]$$

This is quite the complex mess. However, if terms are written with a common denominator and collated it drastically simplifies.

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$$g_{00,2} = \frac{1}{\rho^4 F^2}\Big[8m^2a^4r^2\rho^2\Delta\cos\theta\sin^3\theta + 8m^2a^2r^2\rho^2\cos\theta\sin\theta F$$

$$+ 8m^2a^4r^2\cos\theta\sin^3\theta F + 2a^2\rho^2F^2\cos\theta\sin\theta - 8m^2a^4r^2\rho^2\Delta\cos\theta\sin^3\theta$$

$$- 8m^2a^2r^2F\rho^2\cos\theta\sin\theta + \underline{4ma^2rF^2\cos\theta\sin\theta} - 2a^2\rho^2\cos\theta\sin\theta F^2$$

$$- 8ma^4r^2\cos\theta\sin^3\theta F\Big]$$

This shows many terms cancel. All but the underlined term cancel.

$$g_{00,2} = \frac{4ma^2r\cos\theta\sin\theta}{\rho^4}$$

Putting this together with g^{22}

$$\boxed{\Gamma^2_{00} = -\frac{2ma^2r\cos\theta\sin\theta}{\rho^6}} \quad \square$$

Conclusions: Looking at these results, which have turned out to be quite simple, it becomes obvious that calculations in these particular coordinates may not be the most useful. However, considering the actual results, the simple forms they take seem to hint at the elegant nature of the Kerr solution. The Boyer-Lindquist do present a conviont form, highlighting the frame dragging that is a result of the $\phi - t$ mixing. This helps physical interpretation of the Kerr spacetime.