ADVANCED GENERAL RELATIVITY

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PROBLEM 8.1. : Hamiltonian formulation of Einstein's equations.

The Einstein-Hilbert action is

$$S = \int d^4x \sqrt{-g}\, R = \int d^4x\, N\sqrt{\gamma}\left(\bar{R} + K_{ab}K^{ab} - K^2\right)$$

(where we have neglected surface terms)

with the constraints (coming from $\frac{\delta S}{\delta N} = \frac{\delta S}{\delta N^a} = 0$)

$$\begin{cases} \bar{R} + K^2 - K_{ab}K^{ab} = 0 \\ \nabla_b K_a{}^b - \bar{\nabla}_a K = 0 \end{cases}$$

Gauss-Codazzi equations

Define

$$\dot{\gamma}_{ab} := \partial_t \gamma_{ab} = \mathcal{L}_{\vec{\partial}_t} \gamma_{ab}$$

so then the Lagrangian depends on γ and $\dot{\gamma}$

$$\mathcal{L}(\gamma, \dot{\gamma}) = N\sqrt{\gamma}\left(\bar{R} + K_{ab}K^{ab} - K^2\right)$$

since

$$K_{ab} = \frac{1}{2N}\left(\dot{\gamma}_{ab} - 2\nabla_{(a}N_{b)}\right)$$

and $\bar{R} = \bar{R}(\gamma)$ since $\frac{\partial \bar{R}}{\partial \dot{\gamma}} = 0$, γ is spatial

In order to get the Hamiltonian we perform a Legendre transformation. In analogy with classical mechanics:

$$H(q,p) = \dot{q}p - L(q,\dot{q}) \longleftrightarrow \mathcal{H}(\gamma,\Pi,N,N_a) = \dot{\gamma}_{ab}\Pi^{ab} - \mathcal{L}(\gamma,\dot{\gamma},N,N_a)$$

First, define the canonical momenta as

$$\Pi^{ab} = \frac{\partial \mathcal{L}}{\partial \dot{\gamma}_{ab}} = \sqrt{\gamma}\,(K^{ab} - K\gamma^{ab})$$

and from the former expression for K_{ab} we have

$$\dot{\gamma}_{ab} = 2NK_{ab} + 2\bar{\nabla}_{(a}N_{b)}$$

so then,

$$\mathcal{H} = \dot{\gamma}_{ab}\Pi^{ab} - \mathcal{L} = \sqrt{\gamma}\left[2NK_{ab}K^{ab} - 2NK^2 + 2K^{ab}\bar{\nabla}_{(a}N_{b)} - 2K\gamma^{ab}\bar{\nabla}_{(a}N_{b)}\right] -$$

$$- \sqrt{\gamma}\left[N\bar{R} + NK_{ab}K^{ab} - NK^2\right] =$$

$$= \sqrt{\gamma}\left[-N\bar{R} + NK_{ab}K^{ab} - NK^2 + 2K^{ab}\bar{\nabla}_{(a}N_{b)} - 2K\gamma^{ab}\bar{\nabla}_{(a}N_{b)}\right]$$

Now, we use $\Pi_{ab}\Pi^{ab} = \gamma\left[K_{ab}K^{ab} + (\underbrace{\mathrm{Tr}\,\gamma}_{=d=3} - 2)K^2\right] \Rightarrow K_{ab}K^{ab} = \frac{1}{\gamma}\Pi^{ab}\Pi_{ab} - K^2$

Hence,

$$\mathcal{H} = \sqrt{\gamma}\left[-N\bar{R} + \frac{N}{\gamma}\Pi_{ab}\Pi^{ab} \underbrace{- NK^2 - NK^2}_{-2NK^2} + 2K^{ab}\bar{\nabla}_{(a}N_{b)} - 2K\gamma^{ab}\bar{\nabla}_{(a}N_{b)}\right]$$

Next let us use $\gamma_{ab}\Pi^{ab} = \sqrt{\gamma}\,(1 - \underbrace{\mathrm{Tr}\,\gamma}_{3})K \Rightarrow K = \frac{-1}{2\sqrt{\gamma}}\,\gamma_{ab}\Pi^{ab}$

Thus,

$$\mathcal{L} = \sqrt{\gamma}\Big\{ -N\bar{R} + \frac{N}{\gamma}\Pi_{ab}\Pi^{ab} - 2N\frac{1}{4\gamma}\left(\gamma_{ab}\Pi^{ab}\right)^2 +$$

$$+ 2\underbrace{K^{ab}}\bar{\nabla}_{(a}N_{b)} - 2\left(\frac{-1}{2\sqrt{\gamma}}\right)\underbrace{(\gamma_{ab}\Pi^{ab})(\gamma^{ab}\bar{\nabla}_{(a}N_{b)})}_{3\Pi^{ab}\bar{\nabla}_{(a}N_{b)}}\Big]$$

Also substitute K^{ab} in terms of Π^{ab}

$$K^{ab} = \frac{1}{\sqrt{\gamma}}\Pi^{ab} + K\gamma^{ab} = \frac{1}{\sqrt{\gamma}}\Pi^{ab} - \frac{1}{2\sqrt{\gamma}}(\gamma_{ab}\Pi^{ab})\gamma^{ab} =$$

$$= \frac{-1}{2\sqrt{\gamma}}\Pi^{ab}$$

Hence,

$$\mathcal{L} = \sqrt{\gamma}\Big[N\Big(-\bar{R} + \frac{1}{\gamma}\Pi_{ab}\Pi^{ab} - \frac{1}{2\gamma}\Pi^2\Big) - \frac{1}{\gamma}\Pi^{ab}\bar{\nabla}_{(a}N_{b)} +$$

$$+ \frac{3}{\sqrt{\gamma}}\Pi^{ab}\bar{\nabla}_{(a}N_{b)}\Big] =$$

$$= \sqrt{\gamma}\Big[N\Big(-\bar{R} + \frac{1}{\gamma}\Pi_{ab}\Pi^{ab} - \frac{1}{2\gamma}\Pi^2\Big) + \frac{2}{\sqrt{\gamma}}\Pi^{ab}\bar{\nabla}_{(a}N_{b)}\Big]$$

The last term can be rewritten as

$$\frac{2}{\sqrt{\gamma}}\Pi^{ab}\bar{\nabla}_{(a}N_{b)} = \frac{1}{\sqrt{\gamma}}\Pi^{ab}\left(\bar{\nabla}_a N_b + (a \leftrightarrow b)\right) =$$

$$= \frac{1}{\sqrt{\gamma}}\Big(\underbrace{\bar{\nabla}_a(\Pi^{ab}N_b)}_{\text{surface term}} - N_b\bar{\nabla}_a\Pi^{ab} + (a \leftrightarrow b)\Big) =$$

$$= -\frac{2}{\sqrt{\gamma}}N_b\bar{\nabla}_a\Pi^{ab}$$

Finally the Hamiltonian turns out to be

$$\mathcal{H} = \sqrt{\gamma}\left[N\left(-\bar{R}+\frac{1}{\gamma}\Pi_{ab}\Pi^{ab}-\frac{1}{2\gamma}\Pi^{2}\right)-\frac{2}{\sqrt{\gamma}}N_b\bar{\nabla}_a\Pi^{ab}\right]$$

So the action can be rewritten as

$$S=\int d^4x\; \dot{\gamma}_{ab}\Pi^{ab}-N\overbrace{\left[\sqrt{\gamma}\left(-\bar{R}+\frac{1}{\sqrt{\gamma}}\Pi_{ab}\Pi^{ab}-\frac{1}{2\gamma}\Pi^{2}\right)\right]}^{:=\tilde{\mathcal{H}}} -$$

$$-N_b\underbrace{\left[-2\,\bar{\nabla}_a\Pi^{ab}\right]}_{:=\tilde{\mathcal{H}}^{b}}$$

where the total Hamiltonian is given by

$$\mathcal{H}=N\tilde{\mathcal{H}}+N_b\tilde{\mathcal{H}}^{b}$$