# Advanced General Relativity

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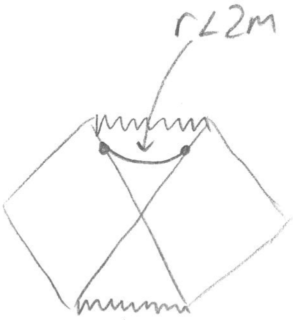
## Problem 7.4.

: In the interior of the Schwarzschild black hole consider spatial hypersurfaces $r = r_0$ where $r_0 < 2m$ is a constant.




$\exists$ maximal hypersurface $(K=0)$ ?

$$ds^2 = -\left(1-\frac{2m}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2m}{r}} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

We want to look for a choice of $r_0$ such that $K = \gamma^{ij}K_{ij} = 0$

The metric in the region where $r_0 < 2m$ :

$$ds^2 = \left(\frac{2m}{r_0} - 1\right)dt^2 + r_0^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

Choosing our induced coordinates as $\begin{cases} t = t(u) = u^1 \\ r = r_0 = \text{const.} \\ \theta = \theta(u) = u^2 \\ \phi = \phi(u) = u^3 \end{cases}$

the induced metric results :

$$\gamma_{ij} = \left(\frac{2m}{r_0} - 1\right) t_{ij} + r_0^2\left(\theta_{ij} + \sin^2\theta\, \phi_{ij}\right)$$

where the matrix $(t)_{ij}$ is defined as :

$$t_{ij} := t_{,i}\,t_{,j} = \frac{\partial t}{\partial u^i}\frac{\partial t}{\partial u^j} = \begin{pmatrix} \left(\frac{\partial t}{\partial u^1}\right)^2 & \left(\frac{\partial t}{\partial u^1}\frac{\partial t}{\partial u^2}\right) & \left(\frac{\partial t}{\partial u^1}\frac{\partial t}{\partial u^3}\right) \\ \left(\frac{\partial t}{\partial u^2}\frac{\partial t}{\partial u^1}\right) & \left(\frac{\partial t}{\partial u^2}\right)^2 & \left(\frac{\partial t}{\partial u^2}\frac{\partial t}{\partial u^3}\right) \\ \left(\frac{\partial t}{\partial u^3}\frac{\partial t}{\partial u^1}\right) & \left(\frac{\partial t}{\partial u^3}\frac{\partial t}{\partial u^2}\right) & \left(\frac{\partial t}{\partial u^3}\right)^2 \end{pmatrix}$$

and similarly for $\theta_{ij}$, $\phi_{ij}$.

Let us choose $n_a \sim \nabla_a r$ as a normal vector since the choice of $\nabla_a t$ is trivial ( $\partial_t$ is a Killing vector).

$K_{ij}{}^a = (+1) K_{ij} n^a$ since it's a spacelike hypersurface and thus $\vec{n} \cdot \vec{n} = +1 \Longrightarrow n_a = \frac{1}{\sqrt{1-\frac{2m}{r_0}}} \nabla_a r$

We need to compute $K_{ij}(n)$

$$K_{ij}(n) = -n_a \left( \underbrace{\cancel{\frac{\partial^2 x^a}{\partial u^i \partial u^j}}}_{\to 0} + \Gamma^a{}_{bc} \frac{\partial x^b}{\partial u^i} \frac{\partial x^c}{\partial u^j} \right) = \frac{-1}{\sqrt{1-\frac{2m}{r_0}}} \left( \Gamma^r{}_{ab} \left(X^{ab}\right)_{ij} \right)$$

because of our choice of coordinates

Since $n_a \sim \nabla_a r$ the only relevant Christoffel symbols are

$$\begin{cases} \Gamma^r_{tt} = \frac{m}{r^2}\left(1-\frac{2m}{r}\right) \\ \Gamma^r_{rr} = -\frac{1}{r^2\left(1-\frac{2m}{r}\right)} \\ \Gamma^r_{\theta\theta} = -r\left(1-\frac{2m}{r}\right) \\ \Gamma^r_{\phi\phi} = -r\left(1-\frac{2m}{r}\right)\sin^2\theta \end{cases}$$

$$K_{ij}(n) = \frac{-1}{\sqrt{1-\frac{2m}{r_0}}} \left( \Gamma^r_{tt}\, t_{ij} + \Gamma^r_{rr} \underbrace{\cancel{r_{ij}}}_{\to 0 \text{ since } r = r_0 = \text{const.}} + \Gamma^r_{\theta\theta}\, \theta_{ij} + \Gamma^r_{\phi\phi}\, \phi_{ij} \phi_{ij} \right) =$$

where $t_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $r_{ij} = 0$, $\theta_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\phi_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$= \frac{-1}{\sqrt{1-\frac{2m}{r_0}}} \left[ \frac{m}{r_0^2}\left(1-\frac{2m}{r_0}\right) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - r_0\left(1-\frac{2m}{r_0}\right) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} - r_0\left(1-\frac{2m}{r_0}\right)\sin^2\theta \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right]$$

$$= \frac{-1}{\sqrt{1-\frac{2m}{r_0}}} \begin{pmatrix} \frac{m}{r_0^2}\left(1-\frac{2m}{r_0}\right) & & \\ & -r_0\left(1-\frac{2m}{r_0}\right) & \\ & & -r_0\left(1-\frac{2m}{r_0}\right)\sin^2\theta \end{pmatrix}$$

The induced metric in explicit matrix form is

$$\gamma_{ij} = \left(\frac{2m}{r_0}-1\right)\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix} + r_0^2\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix} + r_0^2\sin^2\theta\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix} =$$

$$= \begin{pmatrix}\frac{2m}{r_0}-1 & & \\ & r_0^2 & \\ & & r_0^2\sin^2\theta\end{pmatrix}$$

Hence, $$\gamma^{ij} = \begin{pmatrix}\frac{-1}{1-\frac{2m}{r_0}} & & \\ & \frac{1}{r_0^2} & \\ & & \frac{1}{r_0^2\sin^2\theta}\end{pmatrix}$$

$$K=\gamma^{ij}K_{ij}=\frac{-1}{\sqrt{1-\frac{2m}{r_0}}}\mathrm{Tr}\left[\begin{pmatrix}\frac{-1}{1-\frac{2m}{r_0}} & & \\ & \frac{1}{r_0^2} & \\ & & \frac{1}{r_0^2\sin^2\theta}\end{pmatrix}\begin{pmatrix}\frac{m}{r_0^2}\left(1-\frac{2m}{r_0}\right) & & \\ & -r_0\left(1-\frac{2m}{r_0}\right) & \\ & & -r_0\left(1-\frac{2m}{r_0}\right)\sin^2\theta\end{pmatrix}\right]$$

$$= -\frac{1}{\sqrt{1-\frac{2m}{r_0}}}\left[-\frac{m}{r_0^2} - \frac{2}{r_0}\left(1-\frac{2m}{r_0}\right)\right]$$

$$K=0 \Rightarrow \frac{3m}{r_0^2} - \frac{2}{r_0} = 0 \Rightarrow \underline{r_0 = \frac{3}{2}m}$$