FK8024 Special Topics in Theoretical Physics

Nader Stenberg

Problem 4.3

Problem description

In the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \frac{1}{1 - \frac{2m}{r}}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2), \tag{1}$$

we want to make the substitutions

$$\tau = \frac{2^{1/2}}{3}\frac{1}{m^{1/2}}r^{3/2}, \tag{2}$$

$$\rho^2 = \frac{3^{4/3}}{2^{2/3}}m^{2/3}\theta^2, \tag{3}$$

$$z = \frac{2^{2/3}}{3^{1/3}}m^{1/3}t, \tag{4}$$

and then take the limit $m \to \infty$, to see what happens.

Solution

We start by getting the differentials of our new variables:

$$d\tau = \left(\frac{r}{2m}\right)^{1/2}dr, \tag{5}$$

$$d\rho = \frac{3^{2/3}}{2^{1/3}}m^{1/3}d\theta, \tag{6}$$

$$dz = \frac{2^{2/3}}{3^{1/3}}m^{1/3}dt. \tag{7}$$

With

$$dr^2 = \frac{2m}{r}d\tau^2, \tag{8}$$

$$r^2 d\theta^2 = \tau^{\frac{4}{3}}d\rho^2, \tag{9}$$

$$dt^2 = \frac{r}{2m}\tau^{-\frac{2}{3}}dz^2, \tag{10}$$

Our metric now becomes

$$ds^2 = -\left(\left(\frac{3}{2^2m}\right)^{\frac{2}{3}} - \frac{1}{\tau^{-2/3}}\right)dz^2 + \frac{2m}{r-2m}d\tau^2 + \tau^{\frac{4}{3}}d\rho^2 + \tau^{\frac{4}{3}}\left(\frac{3^2m}{2}\right)^{\frac{2}{3}}sin^2\left(\frac{2}{3^2m}\right)^{\frac{1}{3}}d\phi^2. \tag{11}$$

Now, in the limit $m \to \infty$, our metric becomes

$$ds^2 = -d\tau^2 + \tau^{-2/3}dz^2 + \tau^{4/3}(d\rho^2 + d\phi^2). \tag{12}$$

We thus find that, in the limit $m \to \infty$, the metric, in our new coordinates, will only be a function of our new "time" coordinate τ, and the only singularity will be as $\tau \to 0$.