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Advanced General Relativity

Problem 3.4.

: Show that (1+1)-Misner space has closed spacelike curves whose normal is timelike.

From (1+1)-Minkowski space $ds^2=-dt^2+dx^2$ Misner space is obtained by identifying every pair of spacetime points connected by a constant boost: $\mathbb{R}^{1,1}/\underbrace{boost}_{\text{Lorentz subgroup } A^n \text{ where } A \text{ maps:}}$

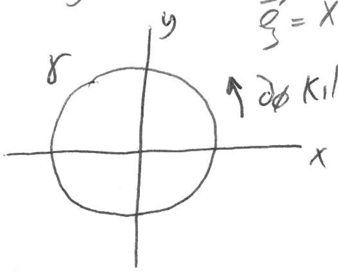
$(t,x)\to(\tilde{t},\tilde{x})=(\tilde{t}\cosh\lambda+\tilde{x}\sinh\lambda,\ \tilde{x}\cosh\lambda+\tilde{t}\sinh\lambda)$

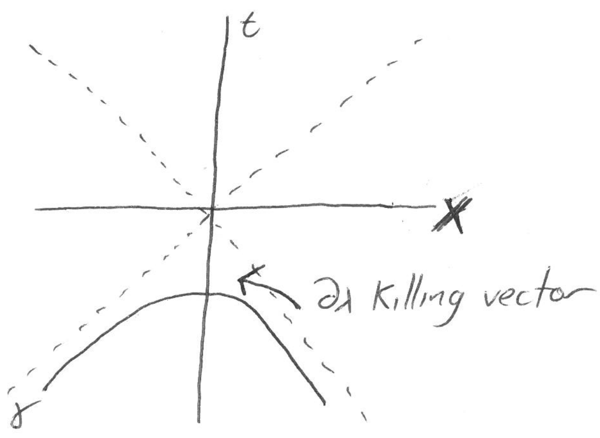
where λ is contant (boost parameter)

The identification of the points is:

$(\tilde{t}\cosh n\lambda+\tilde{x}\sinh n\lambda,\ \hat{x}\cosh n\lambda+\hat{t}\sinh n\lambda)\ \cancel{\forall}\ n\in\mathbb{Z}$

In analogy to a rotation of an angle ϕ in $\mathbb{R}^2$ a good coordinate system is: $\vec{\xi}=x\partial_t-t\partial_x=\partial_\phi$





$x^2+y^2=1\to\gamma(\phi)=\begin{cases}x=\cos\phi\\ y=\sin\phi\end{cases}$

$\begin{cases}x=r\cos\phi\\ y=r\sin\phi\end{cases}\xrightarrow{\quad} r=1$

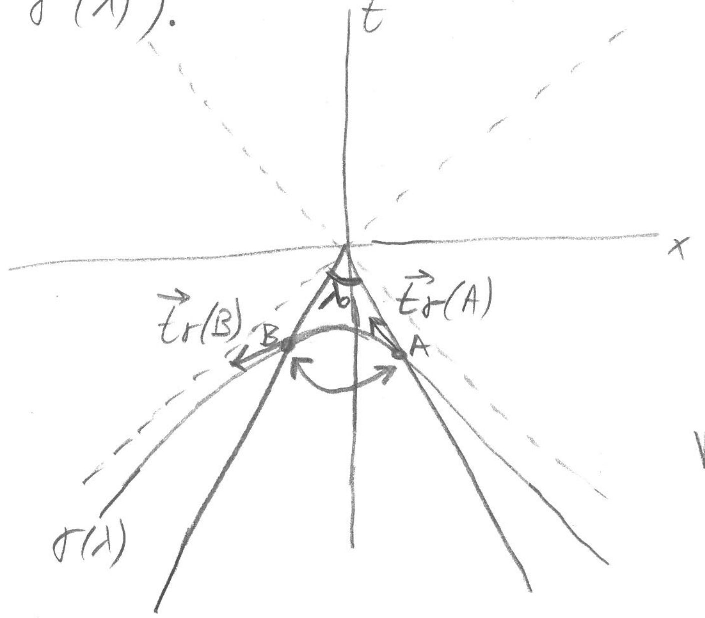
$ds^2=dx^2+dy^2=dr^2+r^2d\phi^2$

$t^2-x^2=1$

$\begin{cases}t=\tau\cosh\lambda\\ x=\tau\sinh\lambda\end{cases}\xrightarrow{\tau=-1}\gamma(\lambda)=\begin{cases}t=-\cosh\lambda\\ x=-\sinh\lambda\end{cases}$

$ds^2=-dt^2+dx^2=-d\tau^2+\tau^2d\lambda^2$

First of all we check that the tangent vector remains the same after identifying the point (i.e. it is paralell transported along $\gamma(\lambda)$).



Boost: $\lambda \to \lambda + \lambda_0$

$\lambda_B = \lambda_A + \lambda_0$

Working in the coordinates (t, x) we have

$$\vec{t} = \frac{dx^a}{d\lambda} = \begin{pmatrix} -\sinh\lambda \\ -\cosh\lambda \end{pmatrix} = \begin{pmatrix} x \\ t \end{pmatrix}$$

$$\begin{pmatrix} t_B \\ x_B \end{pmatrix} = \begin{pmatrix} \cosh\lambda_0 & \sinh\lambda_0 \\ \sinh\lambda_0 & \cosh\lambda_0 \end{pmatrix} \begin{pmatrix} t_A \\ x_A \end{pmatrix}$$

$$\vec{t}_\gamma(A) = \frac{dx^a(A)}{d\lambda} = \begin{pmatrix} -\sinh\lambda_A \\ -\cosh\lambda_A \end{pmatrix}$$

Now we perform a boost on $\vec{t}_\gamma(A)$

$$\begin{pmatrix} \cosh\lambda_0 & \sinh\lambda_0 \\ \sinh\lambda_0 & \cosh\lambda_0 \end{pmatrix} \begin{pmatrix} -\sinh\lambda_A \\ -\cosh\lambda_A \end{pmatrix} = \begin{pmatrix} -\cosh\lambda_0 \sinh\lambda_A - \sinh\lambda_0 \cosh\lambda_A \\ -\sinh\lambda_0 \sinh\lambda_A - \cosh\lambda_0 \cosh\lambda_A \end{pmatrix} =$$

$$= \begin{pmatrix} -\sinh(\lambda_A + \lambda_0) \\ -\cosh(\lambda_A + \lambda_0) \end{pmatrix}$$

which clearly coincides with $\vec{t}_\gamma(B)$ since

$$\vec{t}_\gamma(B) = \begin{pmatrix} -\sinh\lambda_B \\ -\cosh\lambda_B \end{pmatrix} = \begin{pmatrix} -\sinh(\lambda_A + \lambda_0) \\ -\cosh(\lambda_A + \lambda_0) \end{pmatrix}$$

so after the identification the curve γ is well defined every where.

We can easily check that $\gamma(\lambda)$ is a spacelike curve. Working in the new coordinates (τ, λ) the curve γ is parametrised as $x^a = \begin{pmatrix} -1 \\ \lambda \end{pmatrix}$ and thus the tangent vector is

$$t^a = \frac{dx^a}{d\lambda} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

In the coordinates (τ, λ) the metric is given by

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & \tau^2 \end{pmatrix} \overset{\text{evaluated along } \gamma(\lambda)\ (\tau=-1)}{=} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

so the norm of the tangent vector is

$$\vec{t} \cdot \vec{t} = (0 \ 1) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1 > 0 \Rightarrow \underline{\text{spacelike } \forall \lambda}$$

Now we compute the principal normal from the first Frenet equation assuming a constant curvature κ_1:

$$\dot{t}^a = \frac{dt^a}{d\lambda} + \Gamma^a_{bc} t^b t^c = \kappa_1 n^a$$

For this metric the only non-vanishing Christoffel symbols are

$$\Gamma^\tau_{\lambda\lambda} = \tau = -1$$

$$\Gamma^\lambda_{\tau\lambda} = \Gamma^\lambda_{\lambda\tau} = \frac{1}{\tau} = -1$$

$$\kappa_1 n^\tau = 0 + (-1)(t^\lambda)^2 = -1$$

$$\kappa_2 n^\lambda = 0 + 2(-1) t^\tau t^\lambda = 0$$

$$\Rightarrow \kappa_1 \vec{n} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \Rightarrow \kappa_1 = 1$$

and $\vec{n} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

Finally, computing the norm of $\vec{n}$ we have

$$(-1 \ 0) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} = -1 < 0 \Rightarrow \underline{\text{timelike } \forall \lambda}$$

Note we chose to work in the coordinates (τ, λ) and not in (t, x) to avoid the algebra with hyperbolic functions.