One possible solution for Problem 2

Special topics in theoretical physics – FK8024

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As already said by Ingemar during the first lecture, it is enough to open Wald's book [1] and see what he does, which is the following.

Electromagnetic waves in Minkowski space. Let's open [1] and go to page 64. Read one page and a half. Then you know that, in absence of external electromagnetic currents ($j^a = 0$), we can consider an electromagnetic wave with constant amplitude in Minkowski spacetime, having the following form,

$$A_a = C_a \mathrm{e}^{i\phi}. \tag{1}$$

Note that Wald uses S for the phase of the wave, which I do not like, so I will use ϕ. C_a is a constant 1-form (covector), i.e. it is a 1-form with constant norm and always parallel to itself.

The Maxwell equations in the Lorentz gauge tell us that,

$$\partial_a \phi \, \partial^a \phi = 0, \tag{2}$$

and differentiating this we get,

$$(\partial^a \phi) \, \partial_a (\partial_b \, \phi) = 0. \tag{3}$$

Now, the *wave vector field*, by definition, is the vector field which is orthogonal to the surfaces of constant phase ϕ, therefore it is $k^a := \eta^{ab} \partial_b \, \phi$. Equation (2) tells us that k^a is a null vector, and (3) tells us that k^a is a vector field tangent to a family of geodesics. Hence, if we follow k^a, we follow null geodesics.

Remainder 1. A geodesic $\gamma(\tau)$, by definition, is a curve whose tangent vector t^a is parallel to itself when transported along $\gamma(\tau)$ [1, eq. 3.3.1], i.e.

$$t^a \nabla_a t^b = 0, \quad \text{or equivalently} \quad \nabla_{\dot\gamma} \dot\gamma = 0. \tag{4}$$

In our case, (3) is exactly (4) for k^a.

At this point, we define the *frequency* of the wave observed by an observer with 4-velocity u^a, as minus the rate of change of the phase of the wave, projected on u^a [1, eq. 4.2.38],

$$\omega := -u^a \partial_a \phi = -u^a k_a. \tag{5}$$

The redshift factor. Our metric is

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + F(t-z)^2\mathrm{d}x^2 + G(t-z)^2\mathrm{d}y^2 + \mathrm{d}z^2, \tag{6}$$

with $F(t-z) = 1 + (t-z)\Theta(t-z)$ and $G(t-z) = 1 - (t-z)\Theta(t-z)$, and $\Theta(x)$ the Heaviside distribution. For $t \neq z$, this metric is nothing but the Minkowski metric, written in Rosen coordinates.

We have two electromagnetic waves traveling in the direction x and y. The procedure is the same for both waves, so we will make the explicit computations only for the one traveling in the x direction. Then, the wave vector point in the x direction.

Now we notice that, at the event of emission P_1 and at the event of detection P_2 of the wave, we are in Minkowski space, and therefore we have ten Killing vector fields (namely, the generators of the Poincaré group). One of them is $\xi^a = (\partial_x)^a$, the generator of translations along the x direction (which is also one of our basis vectors).

Remainder 2. As explained in [1, Appendix C], a Killing vector field ξ^a is defined as a vector field whose integral curve is a path along which the metric is *invariant*, i.e. the diffeomorphism induced by the Killing vector field is an isometry for the metric. In formulae,

$$\mathscr{L}_{\xi^c}\, g_{ab} = 2\nabla_{(a}\xi_{b)} = 0, \tag{7}$$

where $\nabla_{(a}\xi_{b)} = \dfrac{1}{2}\left(\nabla_a\xi_b + \nabla_b\xi_a\right)$ and $\mathscr{L}_{\xi^c}$ is the Lie derivative along ξ^c.

When a Killing vector field is present (i.e., a symmetry is present), a quantity is conserved along the geodesic motion. If the tangent vector to the geodesic is t^a, then $t^a\xi_a = \text{const.}$ is conserved [1, Proposition C.3.1, p. 442]. In our case, this quantity reads $k_a\xi^a$. Let's keep this in mind for later.

We want to compute the redshift factor of the electromagnetic wave, so we need the frequency. We know that it is $\omega = -k_a u^a$, with u^a 4-velocity of the observer. Now we suppose that the observer (the detector) is at rest, so that $u^{\mathbf{a}} = (1,0,0,0)$.[1] Then, the projection of k^a onto u^a must be minus the projection of k^a onto the (x,y,z) hyperplane, because k^a is a null vector (i.e., it is inclined by 45 degrees with respect to the hyperplane (x,y,z) and the t axis; we are in natural units). In formula,

$$k_a u^a = \operatorname{proj}_{u^a}(k^a) = -\operatorname{proj}_{(x,y,z)}(k^a), \tag{8}$$

where the minus sign follows from the fact that we are in Minkowski space, not Euclidean. On the other hand, we know that $k^a \parallel (\partial_x)^a = \xi^a$, so the

[1] Note that the index $\mathbf{a}$ is boldface, following the Penrose's convention. Boldface indices *can* assume numeric values, i.e. they are *not* abstract indices.

projection onto (x, y, z) is,

$$\text{proj}_{(x,y,z)}(k^a) = -\frac{k_a \xi^a}{(\xi^a \xi_a)^{1/2}}, \tag{9}$$

where the normalization of the Killing vector field follows from the definition of projection itself.

Now we have an expression for the frequency of the wave,

$$\omega = -k_a u^a = \frac{k_a \xi^a}{(\xi^a \xi_a)^{1/2}}. \tag{10}$$

All we need to do is to compute the norm of the Killing vector field ξ^a,

$$(\xi^a \xi_a)^{1/2} = (\eta_{ab} \xi^a \xi^b)^{1/2} = \sqrt{F(t-z)^2} = 1 + (t-z)\Theta(t-z). \tag{11}$$

At this point, we remember that $k_a \xi^a$ is a conserved quantity, so it is the same at the event of emission P_1 and the event of detection P_2. Then, from (10) we find,

$$\left[\omega(\xi^a \xi_a)^{1/2}\right]\Big|_{P_1} = \left[\omega(\xi^a \xi_a)^{1/2}\right]\Big|_{P_2}, \tag{12}$$

which implies,

$$\frac{\omega_{\text{det}}}{\omega_{\text{em}}} = \frac{\left[(\xi^a \xi_a)^{1/2}\right]\big|_{P_1}}{\left[(\xi^a \xi_a)^{1/2}\right]\big|_{P_2}} = \frac{1 + (t_1 - z_1)\Theta(t_1 - z_1)}{1 + (t_2 - z_2)\Theta(t_2 - z_2)}. \tag{13}$$

Then, the redshift factor is easily computed,

$$z := \frac{\lambda_{\text{det}} - \lambda_{\text{em}}}{\lambda_{\text{em}}} = \frac{\omega_{\text{em}}}{\omega_{\text{det}}} - 1 = \frac{1 + (t_2 - z_2)\Theta(t_2 - z_2)}{1 + (t_1 - z_1)\Theta(t_1 - z_1)} - 1. \tag{14}$$

If we define $u := t - z$, then we can write,

$$z = \frac{\omega_{\text{em}}}{\omega_{\text{det}}} - 1 = \frac{1 + u_{\text{det}}\Theta(u_{\text{det}})}{1 + u_{\text{em}}\Theta(u_{\text{em}})} - 1. \tag{15}$$

For the wave in the y direction, all we have to do is to use $G(t-z)$ instead of $F(t-z)$ (the Killing vector field would be ∂_y, and the norm would be different), and find,

$$z = \frac{\omega_{\text{em}}}{\omega_{\text{det}}} - 1 = \frac{1 - u_{\text{det}}\Theta(u_{\text{det}})}{1 - u_{\text{em}}\Theta(u_{\text{em}})} - 1. \tag{16}$$

References

[1] R. Wald, *General relativity* (University of Chicago Press, 2010).