Quasi-local Mass

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Abstract

The notion of quasi-local mass is examined, specifically the definitions suggested by Hawking and Geroch. While these are not fully satisfactory as definitions of quasi-local mass, they have nevertheless proven to be useful tools, for example in proving the positivity of the ADM mass and a version of the Penrose inequality. The mass definitions are evaluated in various special cases, demonstrating explicitly that they can become negative for some very simple surfaces. For a few special spacetimes, a class of surfaces is identified for which the Hawking mass makes sense. Corrections are made to both definitions in the presence of a non-zero cosmological constant. Furthermore, the monotonicity of the Geroch mass under the inverse mean curvature flow (IMCF) is studied in detail, including a numerical evaluation of the evolution of a spheroid.

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Contents

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1 Introduction

In 1915, Albert Einstein published his theory of general relativity, which describes how matter and energy causes spacetime to curve and give rise to gravity. The Einstein field equations,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{1}$$

connects the metric tensor $g_{\mu\nu}$, which represents the geometry of spacetime, with the stress-energy tensor $T_{\mu\nu}$, which represents the matter and energy content of spacetime. These equations may look simple, but this is only because their complicated nature is hidden behind the Ricci tensor $R_{\mu\nu}$, and the scalar curvature R. These quantities are complicated non-linear functions of the metric tensor and its derivatives, which makes the equations very hard to solve.

Prior to this, Newtonian mechanics was the theory that best explained gravitation, by the famous force law

$$F = G\frac{m_1 m_2}{r^2} \tag{2}$$

Where F is the attractive force that objects of mass m_1 and m_2 experience at a distance r from each other, and G is Newton's gravitational constant. In this description, gravity was essentially identical to the electromagnetic force, described by Coulomb's law,

$$F = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2} \tag{3}$$

where q_1 and q_2 are the electric charges of two objects separated by a distance r, and ε_0 is the vacuum permittivity. Notable differences between the two forces is the fact that the electromagnetic force is much stronger, and can be either attractive or repulsive depending on the signs of the charges, while the gravitational force is exclusively attractive.

The theory of electrodynamics, which describes the electromagnetic force completely, was already well established by the time Einstein published his equations. Of central importance is Gauss' law,

$$Q = \int\limits_V \rho_q dV = \varepsilon_0 \oint\limits_{\mathcal{S}} \vec{E} \cdot d\vec{\mathcal{S}} \tag{4}$$

which states that the total amount of charge Q, which is the integral of the charge density ρ_q, inside the closed surface $\mathcal{S}$ is given by an integral of the electric flux over the entire surface. The importance of this law rests with the fact that it defines charge completely in terms of the electric field: it implies that one can uniquely determine the distribution of charge by only regarding the electric field.

The similarities between gravity and electromagnetism made it natural to believe that there should exist a similar expression for gravity, which would give the total amount of mass contained within a closed surface:

$$M = \int\limits_V \rho_m dV = \oint\limits_{\mathcal{S}} \mathcal{F}(g_{\mu\nu}, \partial g_{\mu\nu}) dS \tag{5}$$

for some function $\mathcal{F}$ of the metric and its derivatives. This would work as a 'quasi-local' definition of mass (in the sense that it is only defined for a closed surface) in terms of the geometry of spacetime. However, such an expression has not been found as of yet. As we shall see, the problem seems to rest with the mass density.

Mass has a very special status in all of physics, being present in almost any calculation in one form or another, so it comes as somewhat of a surprise that mass still does not have a satisfactory definition in the scope of general relativity. To understand why its definition is so elusive, we need another pillar of relativity theory: Einstein's mass-energy equivalence principle

$$E = mc^2 \tag{6}$$

which is perhaps the most famous equation of all time. This equation states the fascinating fact that energy and mass are one and the same. This means that mass is a much more complicated subject than the electric charge of electromagnetism. *Everything* has energy, even the gravitational field itself! Thus, to define a gravitational equivalence of Gauss' law, one has to take into account that the field itself has mass.

From a more mathematical perspective, this fact is reflected in the non-linearity of Einstein's field equations. A field that obeys non-linear equations exhibits self-interaction, which makes its behaviour very complicated to describe. On the other hand, fields described by linear equations, such as the electromagnetic field, obey the superposition principle. This means that the field produced by two sources is the sum of the fields that would have been produced by the sources if they were alone. Thus, if one understands how to relate the charge of a simple source like a point charge to its electric field, then the charge of any field configuration can be calculated, in principle. But this does not apply for gravity, since it does not obey the superposition principle.

But the story does not end there. There is another equivalence principle[1], which states that gravitational forces are equivalent to those acting in an accelerating frame of reference. In other words, it is impossible to tell (by local measurements) whether one is standing on the surface of a gravitating body or if the surface is simply accelerating upwards. This, in turn, implies that one can find a reference frame where the gravitational force (in a single point) goes away: the free falling reference frame.[2]

This makes it impossible to connect the gravitational field to a mass density in the traditional sense, since doing so requires attributing a mass to each individidual point in spacetime. If there is a well-defined mass in a single point, then it should cause curvature; but this can always be transformed away.

Taking gravitational radiation into account, another problem appears. Gravitational radiation carries mass with it, but by the above argument it is not possible to determine whether there is gravitational radiation in a single point. Thus, mass may leave a surface through gravitational radiation, but a *local* measurement on the surface will not detect this. In other words, there cannot be a mass flux density on the surface, even though mass may flow out.

Understanding the interplay between the geometry of spacetime and the mass contained within it is evidently very hard. However, it is worth noting that mass actually has a sensible definition in a special case, namely when the spacetime is spherically symmetric. Such a spacetime cannot produce gravitational radiation, so its dynamics are trivial. This makes it possible to define the mass contained inside a round sphere, and this is known as the Misner-Sharp mass[1]. Any more general definition of quasi-local mass should reduce to this (for spherical surfaces) if spacetime is spherically symmetric. In addition to this, there are a few properties which are usually demanded of such a definition, such as [2]

[1] Usually just called 'the equivalence princple', but it comes in many forms, such as the weak and the strong equivalence principles.

[2] But if there really is a gravitational field, then it is impossible to find a frame of reference in which the force goes away in *all* points.

- The mass should be zero in the limit of very small spheres.
- The mass should always be positive.
- For asymptotically flat spacetimes, it should reproduce the total energy content of the entire spacetime, which is well defined in terms of the ADM and Bondi masses.
- If region A contains region B, then the mass of region A should be larger than or equal to the mass of region B. This is usually referred to as monotonicity.

The ADM [3] [4] and Bondi [5] [4] masses will not be treated in any greater detail in this thesis, but we shall mention them briefly here. These are *global* definitions of mass, which can only be defined for spacetime as a whole. This only works if spacetime is asymptotically flat, which essentially means that the curvature of spacetime goes to zero at infinity. The entire spacetime can then be regarded as an isolated system, which makes it possible to define conserved quantities (in the sense of Noether's theorem) for the entire spacetime, one of which is the total energy. This approach cannot be applied to define a quasi-local mass, since a finite region of spacetime is not an isolated system.[3]

At this point, the reason for the name of this thesis should be evident: a *local* definition of mass, which ascribes mass to points in spacetime, is problematic. A *global* definition (ADM), is already known. A *quasi-local* definition, which only ascribes mass to extended regions, is necessary.

Over the years, a few notable attemps have been made to write down a definition of quasi-local mass. Two well-known examples are the Hawking [4] and Geroch [6] masses, which are the basis of study for this thesis. Neither of these fulfill all the above mentioned properties, but they are interesting nevertheless: The Geroch mass, which is a slightly modified version of the Hawking mass, was used to prove the positivity of the ADM mass[4] [6][7] and the Riemannian Penrose inequality [7] [8]. This makes the Hawking mass all the more interesting, since the Geroch mass is not Lorentz invariant (it requires that one chooses how to divide spacetime into space and time), while the Hawking mass is. Other notable examples are the masses suggested by for example Penrose [11] or Wang and Yau [12], but these will not be treated here.

Hopes are that improved understanding of current definitions may give the insight required to find the right definition: understanding the shortcomings of the current framework is a key step towards the development of a more rigid theory. This thesis aims to aid in this matter by evaluating the Hawking and Geroch masses explicitly in some different situations, and drawing a few conclusions about their behaviour. Along the way, the monotonicty property of the Geroch mass is studied in detail, and a few new properties of the Hawking mass are highlighted.

Since mass is such a central concept in all of physics, it is of great importance to understand its role in general relativity. This is crucial if general relativity is to be united with the rest of physics; for example, the law of conservation of energy is taken for granted in almost all areas of physics, but it is not even clear if it applies at all in general relativity. As another example, one of the biggest open problems in physics today is the unison of general relativity with quantum theory. This problem is intricately tied to the non-local nature of mass, and having a working definition of quasi-local mass would probably help towards the development of a unified theory.

[3] Gravity self-interacts, so finite closed surfaces divide spacetime into two regions which interact gravitationally.

[4] The original proof by Geroch [6] was incomplete, though, until Huisken and Ilmanen put the final bits in place [7]. However, this was after Schoen and Yau [9], and Witten [10], had proven the positive energy theorem using other methods.

2 Background

2.1 Prerequisites and notation

In what follows, it is assumed that the reader has an introductory-level understanding of the theory of general relativity, and a firm grasp of tensor analysis and index notation, which will be used extensively throughout. Things like manifolds, metrics, covariant derivatives and Christoffel symbols are assumed to be familiar to the reader. Basic solutions of Einstein's field equations like the Minkowski and Schwarzschild metrics are also assumed to be familiar.

Index naming will follow the convention that Greek indices, like μ and ν, are used to index tensors on four-dimensional spacetime, such as the metric $g_{\mu\nu}$. These indices takes values between 0 and 3, where the zeroeth component is the time component. Sometimes, the corresponding coordinate label will be used when referencing a specific component: g_{rr} means g_{11} if $x^1 = r$.

It is common to use the Latin letters i, j, k to denote spatial indices, but we shall use them instead to index tensors on two-dimensional submanifolds. For example, the coordinates on a two-dimensional surface are labeled $u^i = (u^1, u^2) \equiv (u, v)$.

In places where there is also a three-dimensional hypersurface, tensors on the hypersurface will be labeled by indices a, b, c. Coordinates on a hypersurface are labeled by $y^a = (y^1, y^2, y^3)$.

The Einstein summation convention is in place, which means that repeated indices are summed over:

$$x_\mu x^\mu = \sum_{\mu=0}^{3} x_\mu x^\mu \tag{7}$$

Vectors and covectors are usually treated with no distinction, since these are dual. In places where indices are unwanted, vectors are denoted with an arrow overhead, such as $\vec{n}$.

2.2 Preliminaries

Before we proceed, it will be necessary to cover some general theory on the geometry of surfaces, since many of the following calculations rely heavily on this. Some theorems from differential geometry are also covered in appendix B, since these will be required for a proof later on.

2.2.1 Two-surfaces in spacetime

A two-dimensional surface, or simply two-surface, is not a particularily complicated thing to understand if you live in a three-dimensional world. Unfortunately for us, we shall be considering two-surfaces in spacetime, which is four-dimensional - and one of those dimensions is timelike, which makes things even more confusing. Working with these surfaces takes some getting used to, which is why we will cover their basics first.

There are essentially two ways to define a surface; either on parametric form

$$x^\mu = x^\mu(u^i), \quad \text{for example} \quad \begin{cases} t = 0 \\ r = 1 \\ \theta = u^1 \\ \varphi = u^2 \end{cases} \tag{8}$$

where the parameters $u^i = (u^1, u^2)$, which functions as coordinates on the surface, are mapped to points $x^\mu = (x^0, x^1, x^2, x^3)$ in spacetime. The second way is to define it implicitly, via a pair of equations

$$F_1(x^\mu) = 0 \quad \text{and} \quad F_2(x^\mu) = 0, \quad \text{for example} \quad x^2 + y^2 + z^2 - 1 = 0 \quad \text{and} \quad t = 0 \tag{9}$$

which are satisfied by any point that lies on the surface. Notice how two equations are required to specify a surface in spacetime; each equation removes one degree of freedom. Without the second equation, the above example would define a three-dimensional hypersurface.

Two-surfaces in spacetime are perhaps harder to understand than three-dimensional hypersurfaces, because they have co-dimension two; which means that they have two independent normal directions. In this thesis we shall mostly be concerned with spacelike two-surfaces, which means that they closely resemble our intuitive notion of a two-dimensional surface, but not quite: the way in which they are embedded in spacetime requires some reflection.

A surface of co-dimension two will have two linearly independent normal vectors. For a spacelike surface, which has two spacelike tangent vectors, these may be spacelike, timelike, or null, but not both spacelike or both timelike.

This also means that the normal vectors are not quite as unique as they would have been if the co-dimension was one, since any linear combination of the normal vectors is also a normal vector. In other words, they span a two-dimensional subspace.

If the surface is given on parametric form, the easiest way to find a pair of surface normals is to first find the tangent vectors. These are given by

$$e^\mu{}_i = \frac{\partial x^\mu}{\partial u^i} \tag{10}$$

It is then relatively straight-forward to solve for the normal vectors by setting up the linear system of equations

$$\begin{aligned} e^\mu{}_1 n_\mu &= 0 \\ e^\mu{}_2 n_\mu &= 0 \end{aligned} \tag{11}$$

and solving for n^μ. This system will have two linearly independent solutions, which span the normal space. However, this method often takes a lot of time. If the surface is instead given implicitly, then one may find a pair of normal vectors directly using the gradient operation:

$$n_\mu = \nabla_\mu F_1, \quad t_\mu = \nabla_\mu F_2 \tag{12}$$

One then only has to normalize these to unit length. For spacelike surfaces, it is always possible to find a pair of null vectors, commonly called $\vec{k}_+$ and $\vec{k}_-$, that span the normal space.

$$k_{+\mu} k_+^\mu = 0 \quad \text{and} \quad k_{-\mu} k_-^\mu = 0 \tag{13}$$

All null vectors are orthogonal to themselves, in the sense that their inner product with themselves are zero. Thus, to guarantee that two null vectors are linearly independent, one must choose their product to be non-zero. It is customary to 'normalize' null vectors in such a way that their product with each other is -2:

$$k_+^\mu k_{-\mu} = -2 \tag{14}$$

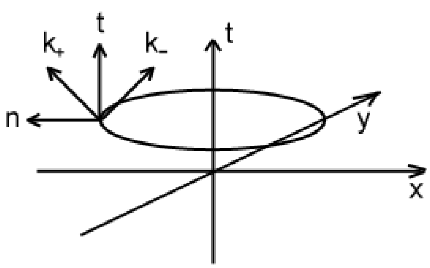

Figure 1: Example: A circle in 2+1 has a timelike normal plane, spanned by the vectors $\vec{n}$ (spacelike) and $\vec{t}$ (timelike), or equivalently, the null vectors $\vec{k}_+$ and $\vec{k}_-$.

This is a convention in the same spirit as the convention to normalize basis vectors to unit length: all formulas used assume this convention.

Note that there is a degree of freedom in the choice of these null normals in the form of a common boost (if the normal space was spacelike, it would be a common rotation). However, this free parameter can be selected freely for our purposes, as it will not affect any of the calculations we will make. A simple choice is to write the null normals as the sum and difference of a timelike and spacelike vector of unit lengths:

$$k_{\pm\mu} = t_\mu \pm n_\mu \quad \text{where} \quad t_\mu t^\mu = -1 \quad \text{and} \quad n_\mu n^\mu = 1 \tag{15}$$

as illustrated in figure 1.

2.2.2 The first and second fundamental forms

A surface $\mathcal{S}$ defined on a manifold $\mathcal{M}$ is a *submanifold* of $\mathcal{M}$. This means that $\mathcal{S}$ inherits a metric γ_{ij} from $\mathcal{M}$, the form of which will depend on the metric of $\mathcal{M}$, $g_{\mu\nu}$, and how $\mathcal{S}$ is embedded in $\mathcal{M}$. This metric is usually called the induced metric or simply surface metric, but is more formally known as the first fundamental form of the surface. As should probably be familiar to the reader, the metric contains information about the intrinsic curvature of the manifold.

To find the induced metric, it is easiest to first express the surface on parametric form. One may then either project the manifold metric $g_{\mu\nu}$ into the tangent space of the surface by contracting it with the tangent vectors of the surface:

$$\gamma_{ij} = g_{\mu\nu} e_i^\mu e_j^\nu \quad \text{where} \quad e_i^\mu = \frac{\partial x^\mu}{\partial u^i} \tag{16}$$

which is equivalent to taking the product of the tangent vectors:

$$\gamma_{ij} = \vec{e}_i \cdot \vec{e}_j \tag{17}$$

or, one may start from the line element ds^2:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{18}$$

and simply substitute the parameterization: $x^\mu = x^\mu(u)$ and $dx^\mu = \frac{\partial x^\mu}{\partial u^i} du^i$. The difference between the two methods is only pragmatic; the latter way is usually easier in practice. For example, one can quite quickly identify the induced metric on the unit sphere starting from the flat 3-space metric

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2 \tag{19}$$

by simply setting $r = 1$ and $dr = 0$.

In contrast to the first fundamental form, the *second* fundamental form contains information about the *extrinsic* curvature of a surface. A very common example used to illustrate curvature is a flat piece of paper rolled into a cylinder. The distance between any two points on the paper (along the paper) does not change when it is rolled, which means that it is intrinsically flat; but something has changed, and this is the extrinsic curvature described by the second fundamental form. In this sense, the second fundamental form tells us how a manifold has been embedded into another manifold, which is why it is sometimes called the shape tensor.

The second fundamental form[5] is usually defined via the Gauss-Weingarten equation, which is easiest stated using a notation without indices: let X and Y be vector fields that are tangent to the surface, but otherwise arbitrary. Then

$$\nabla_X Y = (\nabla_X Y)^\top + (\nabla_X Y)^\perp \quad \text{where} \quad -(\nabla_X Y)^\perp \equiv K(X,Y) \tag{20}$$

where $\nabla_X Y$ is the covariant directional derivative of Y along X. This equation makes a very simple statement: the directional derivative of Y along X can be divided into a component that is tangential to the surface, and a component that is orthogonal to it.

The orthogonal component is called the Weingarten tensor $K(X,Y)$, and is closely related to the second fundamental form. It is a linear function of X and Y, which follows from the linearity of the directional derivative - and this is also what makes it a tensor. It is a rank 3 tensor, because given two vectors X and Y, it gives back a vector that is orthogonal to the surface.

The second fundamental form is then defined as the product of the Weingarten tensor with the normal vector:

$$K(X,Y;n) = \langle K(X,Y), n\rangle = -\langle \nabla_X Y, n\rangle, \tag{21}$$

As a function of X and Y, this is a rank 2 tensor. Given any two vectors X and Y, it will return a scalar which essentially tells us how much the vector field Y must change to stay tangential to the surface as one moves in the direction of X: this is what makes it a measure of extrinsic curvature.

For surfaces of higher co-dimension than one, the normal vector field n is not unique, so the second fundamental form depends on the choice of normal vector. This reflects the fact that such a surface can curve in more than one dimension, which may be illustrated with a simple one-dimensional object like a circle drawn on a piece of paper. The paper is flat, so the circle only curves in a single plane. But if one were to roll the paper, the circle will curve in more than one plane. To answer the question 'what is the curvature of the circle?', one would also need to specify which curvature to measure. This is done mathematically by specifying a normal vector with respect to which the second fundamental form can be calculated.

The expression for the second fundamental form may be brought to a more explicit form by introducing a coordinate basis on the surface and letting the vector fields X and Y be the basis vectors $e_i^\mu = \frac{\partial x^\mu}{\partial u^i}$. One then finds that

$$\begin{aligned} K(\vec{e}_i, \vec{e}_j; \vec{n}) \equiv K_{ij}(\vec{n}) &= -\vec{n}\cdot\nabla_{\vec{e}_i}\vec{e}_j = -n_\mu e_i^\alpha \nabla_\alpha e_j^\mu = -n_\mu e_i^\alpha \left(\frac{\partial e_j^\mu}{\partial x^\alpha} + \Gamma^\mu_{\alpha\beta} e_j^\beta\right) \\ &= -n_\mu \left(\frac{\partial e_j^\mu}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial u^i} + \Gamma^\mu_{\alpha\beta}\frac{\partial x^\alpha}{\partial u^i}\frac{\partial x^\beta}{\partial u^j}\right) = -n_\mu \left(\frac{\partial^2 x^\mu}{\partial u^i \partial u^j} + \Gamma^\mu_{\alpha\beta}\frac{\partial x^\alpha}{\partial u^i}\frac{\partial x^\beta}{\partial u^j}\right) \end{aligned} \tag{22}$$

Given a parameterization of the surface, this expression may be used to calculate the second fundamental form directly.

[5] with respect to a specific choice of normal (if co-dimension is higher than 1).

2.2.3 Foliating spacetime

Given a four-dimensional spacetime, one may single out a spatial hypersurface by fixing the time coordinate. For example, let $\mathcal{M}$ be the Minkowski spacetime

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2 \tag{23}$$

Setting $t = 0$ gives us the flat 3-space described by the metric

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2 \tag{24}$$

This corresponds to the rest space of an inertial observer. This space has a positive definite metric and constitutes a spatial hypersurface, embedded in spacetime. The union of all such hypersurfaces at constant t makes up all of spacetime. More formally, we can define the hypersurfaces Σ_τ as surfaces of constant time, $t = \tau$. Then $\cup_\tau \Sigma_\tau = \mathcal{M}$. Notice how the concept of spacetime now has been separated into space, Σ_τ, and time, τ.

This *foliation* of spacetime can be done in many other ways. For example, one could choose

$$t^2 - r^2 = \tau^2 \tag{25}$$

giving a series of spatial hypersurfaces Σ_τ (which does not correspond to what an inertial observer sees at any instant) that covers spacetime[6], and for any specific τ they define an idea of 'space'.

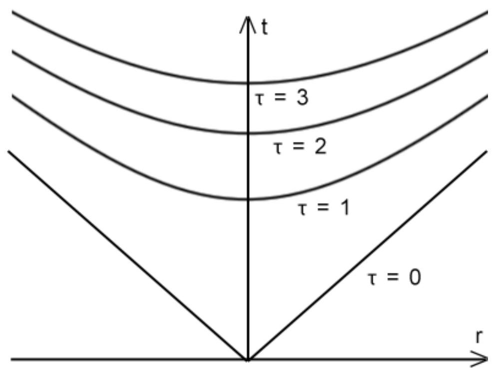

Figure 2: An illustration of the hypersurfaces $t^2 - r^2 = \tau^2$ in the r-t plane. The hypersurface $\tau = 0$ coincides with the light cone.

A foliation of spacetime can be useful for example when formulating Einstein's field equations as an initial value problem: if the metric is specified at some initial hypersurface determined by the parameter τ, then the solution is uniquely determined for all τ.[7] For our purposes, foliations will be required when defining the Geroch mass later on.

[6] Not all of spacetime, though. The case $\tau = 0$ is problematic, because then the hypersurface goes from being spatial to null.

[7] Provided that the hypersurface is a Cauchy hypersurface, which means that all timelike curves in spacetime intersect it once.

2.2.4 Mean extrinsic curvature and the null expansions

We showed above that the second fundamental form may be written as

$$K_{ij}(\vec{n}) = -n_\mu e_i^\alpha \nabla_\alpha e_j^\mu \tag{26}$$

where e_i^α are tangent vectors of the surface $\mathcal{S}$ and n_μ is a normal vector. It is possible to rewrite this as

$$K_{ij}(\vec{n}) = -e_i^\alpha \Big(\underbrace{\nabla_\alpha (n_\mu e_j^\mu)}_{=0} - e_j^\mu \nabla_\alpha n_\mu \Big) = e_i^\alpha e_j^\mu \nabla_\alpha n_\mu \equiv \nabla_i n_j \tag{27}$$

where $\nabla_i n_j$ is to be interpreted as the projection of $\nabla_\mu n_j$ into the tangent space of $\mathcal{S}$. Now, consider the trace of the second fundamental form with respect to the surface metric γ^{ij}:

$$K = \gamma^{ij} K_{ij}(\vec{n}) = \gamma^{ij} \nabla_i n_j = \frac{1}{2} \gamma^{ij} \nabla_{(i} n_{j)} \tag{28}$$

since the metric is symmetric. We recognize the symmetric derivative as the Lie derivative of the metric with respect to a vector:

$$K = \frac{1}{2} \gamma^{ij} \mathcal{L}_{\vec{n}} \gamma_{ij} = \frac{1}{\sqrt{\gamma}} \mathcal{L}_{\vec{n}} \sqrt{\gamma} \qquad \text{where} \quad \gamma \equiv \det(\gamma_{ij}) \tag{29}$$

where the last equality follows from the well known formula for the derivative of the metric determinant, $\partial_\mu \gamma = \gamma \gamma^{\alpha\beta} \partial_\mu \gamma_{\alpha\beta}$. The square root of the metric determinant is the area element $d\mathcal{S}$ of the surface, which means that the trace of the second fundamental form can be interpreted as the change of the area along the normal direction of the surface. In other words, if one was to displace the surface outwards, in such a way that the displacement in each point is along the direction of the normal vector in that point, then the change of the area in that point is proportional to K.[8] In this sense, K is a measure of the extrinsic curvature of the surface. A flat surface will not change at all under a displacement in the normal direction, whereas a heavily curved surface will tend to unfold, as illustrated in the figure on the right.

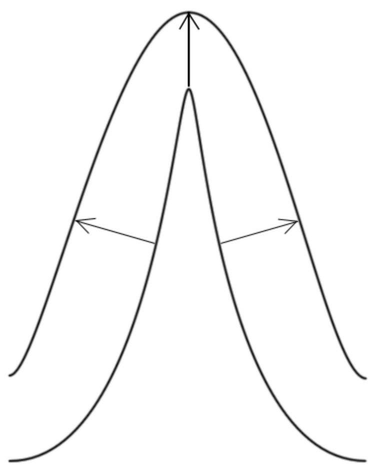

Figure 3: A heavily curved surface (in cross section) unfolding under transport in the normal direction, thus increasing its area.

For a two-surface in four-dimensional spacetime, K will depend on the choice of normals. However, given a foliation of spacetime, there is a natural division of spacetime into space and time components, such that the spacelike normal $\vec{n}$ of a two-surface is unique within the hypersurfaces. It is then common to define the mean extrinsic curvature p of $\mathcal{S}$ as

$$p = \gamma^{ij} K_{ij}(\vec{n}) \tag{30}$$

which is then a measure of spatial curvature in the sense described above. Another common definition is the null expansions $\theta_\pm$, which can be defined given a pair of null normals $\vec{k}_\pm$:

$$\theta_\pm = \gamma^{ij} K_{ij}(\vec{k}_\pm) \tag{31}$$

[8] There is a technicality in the above argument that is worth mentioning. The normal vector field $\vec{n}$ only exists on the surface, which makes both the regular covariant derivative $\nabla \vec{n}$ and the Lie derivative $\mathcal{L}_{\vec{n}}$ problematic, since these require the comparison of nearby points not necessarily on the surface. However, it is always possible to continue the normal vectors outward by taking them to be tangent vectors of geodesics that cross the surface perpendicularly, which allows the derivatives to be defined.

Extending the above argument, we see that these describe the change of the area if one were to displace the surface along the null vectors $\vec{k}_\pm$. As such, they describe the expansion and contraction of wavefronts of light emitted from the surface. However, notice that there is an ambiguity in their definition: the null normals $\vec{k}_\pm$ are not unique, since Lorentz boosting the normal space yields a new pair of normals that are also null. The act of such a Lorentz boost on the null expansions is to make one larger at the cost of the other:

$$\theta_+ \to e^{\alpha}\theta_+ \quad \text{and} \quad \theta_- \to e^{-\alpha}\theta_- \tag{32}$$

This reflects the fact that the amount of expansion and contraction is not an invariant quantity, but depends on the relative motion of the observer. The signs on these quantities are not ambiguous, however, which makes the null expansions useful for example in the analysis of black holes. They can be used to study something called trapped surfaces, which is a surface upon which both null expansions are negative, meaning that all light fronts emitted from the surface are contracting. The existence of such a surface implies, under some assumptions regarding the matter contents, that spacetime will be singular. In other words, they indicate the presence of a black hole.

Finally, we note that the product of the null expansions is an invariant quantity, one that all observers may agree upon, since the ambiguous factor above cancels out.

3 The Hawking Mass

The Hawking mass was introduced by Stephen Hawking in [13]. One of many suggested definitions of quasi-local mass, it is perhaps the most widely known even though it is not completely satisfactory as a definition of mass, as will be shown shortly. It makes sense in some special cases, but not in general. We shall begin by simply stating its definition: Given a closed spacelike two-surface $\mathcal{S}$, the Hawking mass M_H is defined as

$$M_H = \sqrt{\frac{A}{16\pi}}\left(1 + \frac{1}{16\pi}\oint\limits_{\mathcal{S}} \theta_+\theta_- d\mathcal{S}\right) \tag{33}$$

where $\theta_\pm$ are the null expansions of the surface and $A = \text{Area}(\mathcal{S})$. The first observation that one makes is that it has the proper dimensionality to be a mass - length, in natural units. It is also an explicit expression which is very straight-forward to calculate. Another observation that may be made right off the bat is that if one of the null expansions is zero over the whole surface, the whole expression reduces down to $M_H = \sqrt{A/16\pi}$. This is exactly what happens on the event horizon of a stationary black hole. The area of a Schwarzschild black hole is $16\pi M^2$, so the Hawking mass reduces to $M_H = M$ on the horizon. As we shall see shortly, the mass of any spherical surface in the Schwarzschild spacetime will yield the mass $M_H = M$, which is no accident.

The definition of the Hawking mass has its roots in spherically symmetric spacetimes. In such a case, it is possible to define a mass function M through

$$M(r) = \frac{r}{2}(1 - g^{\mu\nu}\nabla_\mu r \nabla_\nu r) \tag{34}$$

this is known as the Misner-Sharp mass [1], and it represents the mass contained within spheres of area radius r. Notice specifically how this implies that

$$g^{rr} = 1 - \frac{2M(r)}{r} \tag{35}$$

if r is used as a coordinate.

It is possible to define the Misner-Sharp mass only in spherically symmetric spacetimes for a few reasons. Firstly, there is a well-defined radius function r, which may be defined in terms of the area of the special round spheres singled out by the rotation group, along which the mass of the spacetime must be distributed. Secondly, there is no gravitational radiation in a spherically symmetric spacetime, so the dynamics of spacetime itself are trivial, eliminating the problem of mass leaving the surface.

In gravitational models of stars, the Misner-Sharp mass turns out to be the integral of the mass density of the star [14]

$$M_{MS} = \int 4\pi\rho(r)r^2 dr \tag{36}$$

as it would have been if space was *flat*. It would be more natural to expect that the mass should be

$$M_{expected} = \int \rho dV = \int \frac{4\pi\rho(r)r^2 dr}{\sqrt{1 - \frac{2M(r)}{r}}} \tag{37}$$

where the curvature of space is taken into account in the volume element. The latter mass is larger, and the discrepancy may in fact be accredited to the gravitational binding energy (which is negative) being included in the Misner-Sharp mass.

The Hawking mass of a sphere in a spherically symmetric spacetime evaluates precisely to the Misner-Sharp mass, so one could say that the Hawking mass is a generalized version of the Misner-Sharp mass. The Misner-Sharp mass cannot be defined when spherical symmetry is absent, but the Hawking mass may be calculated for any closed surface, in any spacetime. This was the general idea idea behind its definition; to extend the Misner-Sharp mass.

The main feature of the Hawking mass is the null expansions. The idea is that the invariant product $\theta_+\theta_-$ contains information about the surface that reflects its mass content: the presence of mass inside the surface affects the behaviour of null geodesics that cross the surface. Assuming this, and demanding that the right answers are produced in the basic cases of Minkowski and Schwarzschild, one arrives at the Hawking mass. Unfortunately, it does not work for general surfaces in general spacetimes. What is clear, however, is that it is a measure which is affected by the presence of mass inside the surface, but in a way that is not completely understood. For this reason, the Hawking mass is still of interest to study.

The Hawking mass fulfills some of the properties that are wanted of a mass definition, but not all. For example, the Hawking mass of spheres go to zero as their geodetic radius goes to zero [29], reflecting the fact that a point should have no mass. If the surface is taken to be a large asymptotic sphere, the Hawking mass equates to the ADM mass. However, it is not always positive, which will be shown explicitly in the following calculations.

3.1 Sphere in Minkowski

To begin with, we shall calculate the Hawking mass of a sphere in the flat Minkowski spacetime. We know a priori that the result is zero, because the Hawking mass was constructed to fulfill this criteria; that spheres in empty space has no mass (but not more general surfaces). Nevertheless, this is a good starting point to illustrate how the Hawking mass is evaluated.

Let $\mathcal{M}$ the the Minkowski spacetime, described in spherical polar coordinates by the line element

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{38}$$

Next, let $\mathcal{S}$ be the two-sphere defined by constant radius $r = R$ and constant time $t = \tau$.[9] A natural parameterization of this surface is

$$\begin{aligned} t &= \tau \\ r &= R \\ \theta &= u = u^1 \\ \varphi &= v = u^2 \end{aligned} \tag{39}$$

where $u^i = (u, v)$ are coordinates on $\mathcal{S}$. The first step in the calculation is to find the null normals of the surface. We will do this by finding a timelike and a spacelike normal vector, which we can then combine to null normals. The simplest way to do this is to express the surface as the level set of two scalar functions

$$\phi = \tau - t \quad \text{and} \quad \psi = r - R \tag{40}$$

[9]We have to specify the time to define a two dimensional surface; otherwise we would be dealing with a three-dimensional hypersurface. This particular choice, $t =$ constant, corresponds to the rest space of an inertial observer.

So that the gradients of these functions give us the normal (co-)vector fields t_μ and n_μ:[10]

$$t_\mu = \frac{\nabla_\mu \phi}{\sqrt{|\nabla_\mu \phi \nabla^\mu \phi|}} \quad \text{and} \quad n_\mu = \frac{\nabla_\mu \psi}{\sqrt{|\nabla_\mu \psi \nabla^\mu \psi|}} \tag{41}$$

which have been normalized to unit length. We find that

$$\nabla_\mu \phi = \frac{\partial \phi}{\partial x^\mu} = -\delta^t_\mu \quad \Longrightarrow \quad t_\mu = -\delta^t_\mu \quad \text{and} \quad \nabla_\mu \psi = \frac{\partial \psi}{\partial x^\mu} = \delta^r_\mu \quad \Longrightarrow \quad n_\mu = \delta^r_\mu \tag{42}$$

A remark on the notation: the index t on δ^t_μ refers to the component of the tensor δ^ν_μ associated with the time-coordinate t. Whenever a coordinate label occurs as an index, it should be taken to mean the component associated with that coordinate. In this case, the gradients came out with unit length, $\nabla_\mu \phi \nabla^\mu \phi = -1$ and $\nabla_\mu \psi \nabla^\mu \psi = 1$, which made the normalization trivial.

We may now construct a pair of future directed null normals that span the normal space of $\mathcal{S}$,

$$k_{\pm\mu} = t_\mu \pm n_\mu = -\delta^t_\mu \pm \delta^r_\mu \quad \Longrightarrow \quad k_{\pm\mu} k^\mu_\pm = 0 \quad \text{and} \quad k_{+\mu} k^\mu_- = -2 \tag{43}$$

We now have everything we need to calculate the second fundamental form, which will give us the null expansions. It is given by

$$K_{ij}(k_\pm) = -k_{\pm\mu} \left(\underbrace{\frac{\partial^2 x^\mu}{\partial u^i \partial u^j}}_{=0} + \Gamma^\mu_{\alpha\beta} \frac{\partial x^\alpha}{\partial u^i} \frac{\partial x^\beta}{\partial u^j} \right) = -k_{\pm\mu} \Gamma^\mu_{\alpha\beta} \delta^\alpha_i \delta^\beta_j = -k_{\mu\pm} \Gamma^\mu_{ij} = \underbrace{\Gamma^t_{ij}}_{=0} \mp \Gamma^r_{ij} \tag{44}$$

since $\frac{\partial x^\mu}{\partial u^i} = \delta^\mu_i$. One only needs to insert the Christoffel symbols, which can be found in any standard reference, such as [15]. This yields

$$K_{ij}(k_\pm) = \pm \begin{pmatrix} R & 0 \\ 0 & R \sin^2 \theta \end{pmatrix} \tag{45}$$

We now need the metric on $\mathcal{S}$, so that we may take the trace of $K_{ij}(k_\pm)$. It is found by setting $r = R$ and $dt = dr = 0$ in the manifold metric:

$$ds^2 = R^2(du^2 + \sin^2 u dv^2) \quad \Longrightarrow \quad \gamma_{ij} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2 \sin^2 \theta \end{pmatrix} \tag{46}$$

allowing us to calculate the null expansions:

$$\theta_\pm = \gamma^{ij} K_{ij}(k_\pm) = \pm \frac{R}{R^2} \pm \frac{R \sin^2 \theta}{R^2 \sin^2 \theta} = \pm \frac{2}{R} \tag{47}$$

which finally yields the Hawking mass:

$$M_H = \sqrt{\frac{A}{16\pi}} \left(1 + \frac{1}{16\pi} \oint_{\mathcal{S}} \theta_+ \theta_- d\mathcal{S} \right) = \sqrt{\frac{4\pi R^2}{16\pi}} \left(1 - \frac{1}{4\pi R^2} \underbrace{\oint_{\mathcal{S}} d\mathcal{S}}_{4\pi R^2} \right) = 0 \tag{48}$$

[10]Notice that the signs of the functions ϕ and ψ have been chosen so that the resulting normal vectors will come out with the right sign: a future-pointing timelike vector and an outward pointing spacelike vector.

3.2 Sphere in Schwarzschild

As stated above, one of the criteria for the definition of the Hawking mass was that it would yield no mass for spheres in an empty spacetime. Another criterion was that in a Schwarzschild spacetime of mass M, a sphere centered around the black hole would yield the mass M. We will now demonstrate this.

Let $\mathcal{M}$ be the Schwarzschild spacetime, described in spherical polar coordinates by the line element

$$ds^2 = -f dt^2 + f^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \quad \text{where} \quad f = \left(1 - \frac{2M}{r}\right) \tag{49}$$

Let $\mathcal{S}$ again be the two-sphere defined by $r = R$ and $t = \tau$, which has the same parameterization in terms of $u^i = (u, v)$ as in Minkowski. The Hawking mass of $\mathcal{S}$ may then be calculated in much the same fashion as in Minkowski, with some minor modifications.

We may use the same functions ϕ and ψ to find timelike and spacelike normals to the surface. Their gradients come out the same, but their lengths are affected by the non-flat metric. We find that

$$t_\mu = \frac{\nabla_\mu \phi}{\sqrt{|\nabla_\mu \phi \nabla^\mu \phi|}} = -\delta^t_\mu f^{1/2} \quad \text{and} \quad n_\mu = \frac{\nabla_\mu \psi}{\sqrt{|\nabla_\mu \psi \nabla^\mu \psi|}} = \delta^r_\mu f^{-1/2} \tag{50}$$

where the factor f from the metric comes in through the normalization. Thus

$$k_{\pm\mu} = t_\mu \pm n_\mu = \begin{pmatrix} -f^{1/2}, & \pm f^{-1/2}, & 0, & 0 \end{pmatrix} \tag{51}$$

The second derivatives $\frac{\partial^2 x^\mu}{\partial u^i \partial u^j}$ vanishes like in Minkowski, so the second fundamental forms may be calculated as

$$K_{ij}(k_\pm) = -k_{\pm\mu}\Gamma^\mu_{\alpha\beta}\delta^\alpha_i\delta^\beta_j = -k_{\mu\pm}\Gamma^\mu_{ij} = f^{1/2}\underbrace{\Gamma^t_{ij}}_{=0} \mp f^{-1/2}\Gamma^r_{ij} \tag{52}$$

Inserting the relevant Christoffel symbols then gives us the second fundamental forms:

$$K_{ij}(k_\pm) = \pm\sqrt{f}\begin{pmatrix} R & 0 \\ 0 & R\sin^2\theta \end{pmatrix} \tag{53}$$

The metric on the surface is the same as in Minkowski, which is easily seen by setting $dt = dr = 0$ in the line element. Using this, one finds the null expansions:

$$\theta_\pm = \gamma^{ij}K_{ij}(k_\pm) = \pm\frac{\sqrt{f}R}{R^2} \pm \frac{\sqrt{f}R\sin^2\theta}{R^2\sin^2\theta} = \pm\frac{2\sqrt{f}}{R} \tag{54}$$

The Hawking mass may now be evaluated:

$$M_H = \sqrt{\frac{A}{16\pi}}\left(1 + \frac{1}{16\pi}\oint_{\mathcal{S}} \theta_+\theta_- d\mathcal{S}\right) = \sqrt{\frac{4\pi R^2}{16\pi}}\left(1 - \frac{f}{4\pi R^2}\underbrace{\oint_{\mathcal{S}} d\mathcal{S}}_{4\pi R^2}\right) = \frac{R}{2}(1-f) \tag{55}$$

Notice that f could be moved out of the integral since it only depends on r, which is constant on the surface. Inserting the expression for f gives

$$M_H = \frac{R}{2}\left(1 - \left(1 - \frac{2M}{R}\right)\right) = M \tag{56}$$

As was expected.

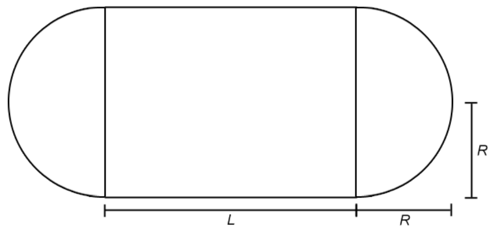

Figure 4: A cross-sectional view of the pill-shaped surface.

3.3 Pill-shaped surface in Minkowski

The Hawking mass was *designed* to work for the previous two examples. For more general surfaces, things don't work out quite as well. In this section and the next, we shall calculate the Hawking mass of some simple non-spherical surfaces in the Minkowski spacetime to show this. As a first example, we shall consider the pill-shaped surface that one gets by joining a cylinder with two hemispheres, as depicted in figure 4 above. Let $\mathcal{S}_C$ be the open cylinder of length L and radius R, given in cylindrical coordinates $x^\mu = (t, r, \theta, z)$ as level surfaces to the functions

$$\phi = \tau - t \quad \text{and} \quad \psi = r - R \tag{57}$$

for $-L/2 \leq z \leq L/2$. This surface may be parameterized as

$$\begin{aligned} &t = \tau \\ &r = R \\ &\theta = u = u^1 \\ &z = v = u^2, \quad -L/2 \leq v \leq L/2 \end{aligned} \tag{58}$$

Furthermore, let $\mathcal{S}_1$ and $\mathcal{S}_2$ be the two hemispheres of radius R that close the cylinder into a pill $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_C \cup \mathcal{S}_2$.

The very first step is to recognize the fact that we may treat the cylinder and the hemispheres separately, and that we already have the null expansions of a sphere. We shall therefore only need to calculate the null expansions of the cylindrical part. In cylindrical coordinates, the Minkowski line element is

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + dz^2 \tag{59}$$

We begin by solving for the timelike and spacelike normal fields:

$$t_\mu = \frac{\nabla_\mu \phi}{\sqrt{|\nabla_\mu \phi \nabla^\mu \phi|}} = -\delta^t_\mu \quad \text{and} \quad n_\mu = \frac{\nabla_\mu \psi}{\sqrt{|\nabla_\mu \psi \nabla^\mu \psi|}} = \delta^r_\mu \tag{60}$$

and the null normals follows as

$$k_{\pm\mu} = t_\mu \pm n_\mu \tag{61}$$

The second fundamental forms of the cylinder are then given by[11]

$$K_{ij}(k_\pm) = -k_{\pm\mu} \Gamma^\mu_{\alpha\beta} \delta^\alpha_i \delta^\beta_j = -k_{\mu\pm} \Gamma^\mu_{ij} = \underbrace{\Gamma^t_{ij}}_{=0} \mp \Gamma^r_{ij} \tag{62}$$

The only relevant non-zero Christoffel symbol is $\Gamma^r_{\theta\theta} = -r$. Thus, one finds that

$$K_{ij}(k_\pm) = \pm \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix} \tag{63}$$

[11] Recall that this formula depends on the fact that $\frac{\partial x^\mu}{\partial u^i} = \delta^\mu_i$, which applies here aswell, but not in general.

Setting $r = R$, $t = \tau$ and $dr = dt = 0$ in the line element yields the surface metric:

$$ds^2 = R^2 d\theta^2 + dz^2 \quad \Longrightarrow \quad \gamma_{ij} = \begin{pmatrix} R^2 & 0 \\ 0 & 1 \end{pmatrix} \tag{64}$$

Thus, the null expansions are

$$\theta_{\pm} = \gamma^{ij} K_{ij}(k_{\pm}) = \pm \frac{1}{R} \tag{65}$$

Now, for the hemispheres, we already know that the null expansions are constant and given by $\theta_{\pm} = \pm \frac{2}{R}$.[12] We thereby have

$$\begin{aligned} M_H &= \sqrt{\frac{A}{16\pi}} \left(1 - \frac{1}{16\pi} \Big(\oint_{\mathcal{S}_1} \frac{4}{R^2} d\mathcal{S}_1 + \oint_{\mathcal{S}_2} \frac{4}{R^2} d\mathcal{S}_2 + \oint_{\mathcal{S}_C} \frac{1}{R^2} d\mathcal{S}_C \Big) \right) \\ &= \sqrt{\frac{A}{16\pi}} \left[1 - \frac{1}{16\pi R^2} \left(\underbrace{4 \Big(\oint_{\mathcal{S}_1} d\mathcal{S}_1 + \oint_{\mathcal{S}_2} d\mathcal{S}_2 \Big)}_{16\pi R^2} + \underbrace{\oint_{\mathcal{S}_C} d\mathcal{S}_C}_{2\pi R L} \right) \right] \\ &= -\sqrt{\frac{A}{16\pi}} \frac{L}{8R} \end{aligned} \tag{66}$$

The total area of the surface is

$$A = 4\pi R^2 + 2\pi R L = 2\pi R(2R + L) \tag{67}$$

So that

$$M_H = -\frac{L}{8R} \sqrt{\frac{R(2R+L)}{8}} = -\frac{L}{16} \sqrt{1 + \frac{L}{2R}} \tag{68}$$

Which is strictly negative, and limits to zero in the case $L \to 0$, when the pill becomes a sphere. This demonstrates the fact that there exists very simple surfaces for which the Hawking mass does not give a satisfactory result. However, as will be shown later on, not all non-spherical surfaces has a negative Hawking mass.

3.4 Spheroid in Minkowski

We will now consider a slightly less trivial example of a surface for which the Hawking mass is negative, namely the axisymmetric ellipsoid, or spheroid. These are divided into two classes: oblate spheroids, which are flattened spheres, and prolate spheroids, which are stretched spheres. The oblate case was presented by D. Hansevi in [16], and we will here extend his calculation to also deal with the prolate case.

The Hawking mass for both classes of spheroids is negative, and goes to zero in the limit when the spheroid approaches a sphere. This will be the first example of a surface which has non-trivial geometry, in the sense that the calculation will be a bit more involved. We shall omit some parts of the calculation that are very similar to the simpler cases treated above.

[12]Notice the fact that the null expansions are discontinuous: they make a sharp jump from $\pm 1/r$ to $\pm 2/r$ at the stichings of the cylinder and the hemispheres. This makes sense, since a sphere has more curvature than a cylinder.

Let $\mathcal{S}$ be the spheroid with lateral axis a and vertical axis c, as defined implicitly by the equations

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} - 1 = 0 \quad \text{and} \quad \tau - t = 0 \tag{69}$$

The lengths of the axes a and c determines whether the spheroid is oblate or prolate: if a is larger than c, then it is oblate, and vice versa. A natural parameterization of this surface is

$$\begin{aligned} t &= \tau \\ x &= a\cos v \sin u \\ y &= a \sin v \sin u \\ z &= c\cos u \end{aligned} \qquad \text{where} \qquad \begin{aligned} 0 &\leq u \leq \pi \\ 0 &\leq v \leq 2\pi \end{aligned} \tag{70}$$

We find the timelike and spacelike normal vector fields

$$t_\mu = -\delta^t_\mu \quad \text{and} \quad n_\mu = N\left(\ 0,\ \ \tfrac{x}{a^2},\ \ \tfrac{y}{a^2},\ \ \tfrac{z}{c^2}\ \right) \quad \text{where} \quad N = \left(\frac{x^2}{a^4} + \frac{y^2}{a^4} + \frac{z^2}{c^4}\right)^{-1/2} \tag{71}$$

Using these, we construct the future directed null normals

$$k_{\pm\mu} = t_\mu \pm n_\mu = \left(\ -1,\ \ \pm\tfrac{Nx}{a^2},\ \ \pm\tfrac{Ny}{a^2},\ \ \pm\tfrac{Nz}{c^2}\ \right) \tag{72}$$

Next, we shall calculate the second fundamental forms:

$$K_{ij}(k_\pm) = -k_{\pm\mu}\left(\frac{\partial^2 x^\mu}{\partial u^i \partial u^j} + \Gamma^\mu_{\alpha\beta}\frac{\partial x^\alpha}{\partial u^i}\frac{\partial x^\beta}{\partial u^j}\right) \tag{73}$$

For the sphere, we were able to parameterize the surface rather trivially by using spherical coordinates for spacetime. This meant that the second derivatives, $\frac{\partial^2 x^\mu}{\partial u^i \partial u^j}$, were all zero, which made the second fundamental forms simple to evaluate. The spheroid, however, does not look particularily simple in neither spherical coordinates nor cartesian coordinates, but cartesian coordinates has the benefit that the Christoffel symbols are all zero. Then one only has to evaluate the second derivatives:

$$K_{ij}(k_\pm) = -k_{\pm\mu}\frac{\partial^2 x^\mu}{\partial u^i \partial u^j} \tag{74}$$

Doing so gives

$$\begin{aligned} K_{uu} &= \pm N\left(\tfrac{x^2}{a^2} + \tfrac{y^2}{a^2} + \tfrac{z^2}{c^2}\right) = \pm N, \quad K_{uv} = 0 \\ K_{vv} &= \pm N\left(\tfrac{x^2}{a^2} + \tfrac{y^2}{a^2}\right) = \pm N\sin^2 u, \quad K_{vu} = 0 \end{aligned} \quad \Longrightarrow \quad K_{ij}(k_\pm) = \pm N\begin{pmatrix} 1 & 0 \\ 0 & \sin^2 u \end{pmatrix} \tag{75}$$

The induced metric on $\mathcal{S}$ is

$$\gamma_{ij} = g_{\mu\nu}\frac{\partial x^\mu}{\partial u^i}\frac{\partial x^\nu}{\partial u^j} = \begin{pmatrix} a^2\cos^2 u + c^2 \sin^2 u & 0 \\ 0 & a^2\sin^2 u \end{pmatrix} \tag{76}$$

So that the null expansions become

$$\theta_\pm = \gamma^{ij}K_{ij}(k_\pm) = \pm N\left(\frac{1}{a^2\cos^2 u + c^2\sin^2 u} + \frac{1}{a^2}\right) \tag{77}$$

At this point, we note that

$$N^2 = \left(\frac{x^2}{a^4} + \frac{y^2}{a^4} + \frac{z^2}{c^4}\right)^{-1} = \left(\frac{\sin^2 u}{a^2} + \frac{\cos^2 u}{c^2}\right)^{-1} = \frac{a^2c^2}{a^2\cos^2 u + c^2\sin^2 u} \tag{78}$$

To simplify things, we set $a^2\cos^2 u + c^2 \sin^2 u = R^2$. Then

$$N^2 = \frac{a^2c^2}{R^2} \quad \text{and} \quad \theta_\pm = \pm ac\left(\frac{1}{R^3} + \frac{1}{a^2R}\right) \tag{79}$$

Now, the Hawking mass may be calculated:

$$M_H = \sqrt{\frac{A}{16\pi}}\left(1 - \frac{a^2c^2}{16\pi}\oint_{\mathcal{S}}\left(\frac{1}{R^6} + \frac{1}{a^4R^2} + \frac{2}{a^2R^4}\right)d\mathcal{S}\right) \tag{80}$$

Solving the above integral, and also finding the area of the spheroid, is a straight forward but long calculation which we shall not present here.[13] The area of the spheroid is

$$A = 2\pi a^2\left(1 + \frac{\mathcal{F}(\chi)}{\chi\sqrt{|\chi^2 - 1|}}\right) \quad \text{where} \quad \mathcal{F}(\chi) \equiv \begin{cases} \operatorname{arccosh}(\chi) & \text{for } \chi \geq 1 \\ \arccos(\chi) & \text{for } \chi < 1 \end{cases} \tag{81}$$

and where we have introduced the dimensionless parameter $\chi = a/c$, which tells us how oblate or prolate the spheroid is. Evaluating the remaining integrals, one finds that

$$M_H = -\frac{a\sqrt{2}}{16}\sqrt{1 + \frac{\mathcal{F}(\chi)}{\chi\sqrt{|\chi^2 - 1|}}}\left(\frac{2\chi^2 - 5}{3} + \frac{\mathcal{F}(\chi)}{\chi\sqrt{|1 - \chi^2|}}\right) \tag{82}$$

If we choose to express the Hawking mass in terms of the area, then the expression becomes a lot simpler:

$$M_H = -\sqrt{\frac{A}{16\pi}}\left(\frac{2\chi^2 - 8}{3} + \frac{A}{2\pi a^2}\right) \leq 0 \tag{83}$$

For example, one can see directly that the Hawking mass goes to zero in the limit when the spheroid becomes a sphere, since then $\text{Area}(\mathcal{S}) = 4\pi a^2$ and $\chi = 1$. The Hawking mass as a function of the parameter χ is plotted in figure 5, illustrating how it is exclusively negative for both oblate and prolate spheroids.

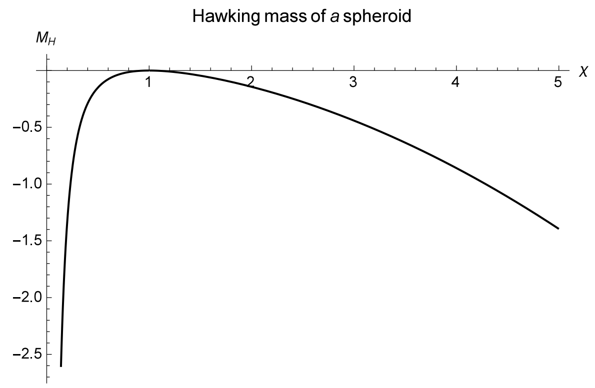

Figure 5: Plot of the Hawking mass of a spheroid. The mass is given in units of length (natural units) in terms of multiples of the lateral axis a. The parameter χ is dimensionless and specifies the oblaticity of the spheroid. $\chi > 1$ corresponds to an oblate spheroid, and $\chi < 1$ corresponds to a prolate spheroid.

[13] A short comment on how this is done: the area element is $d\mathcal{S} = \sqrt{\det(\gamma)}d^2u = Ra\sin u\, du\, dv$. Therefore, all integrals one has to solve involve odd powers of R, which are irrational functions. These are easiest solved by substituting $\cos u = x$, which puts the integral on a standard form that can be solved or looked up in a book on integrals. One then only has to massage the resulting expressions until they look nice.

4 The Geroch Mass

The Geroch mass is a modified version of the Hawking mass, and was originally introduced by Geroch to prove the positive energy theorem [6], which states that the globally defined ADM mass is positive. We shall begin by stating the definition of the Geroch mass: given a closed spacelike two-surface $\mathcal{S}$, the Geroch mass M_G is defined as

$$M_G = \sqrt{\frac{A}{16\pi}}\left(1 - \frac{1}{16\pi}\oint_{\mathcal{S}} p^2 d\mathcal{S}\right) \tag{84}$$

where $p = \gamma^{ij}K_{ij}(\vec{n})$ is the mean extrinsic curvature and $A = \text{Area}(\mathcal{S})$. This requires the choice of a foliation of spacetime, since p is only uniquely defined within a spacelike hypersurface, where $\mathcal{S}$ has a unique spacelike normal vector $\vec{n}$.

Geroch's proof of the positive energy theorem utilizes the fact that this mass definition is monotonely increasing[14] if the surfaces are evolved outwards in a very specific way, namely the 'inverse mean curvature flow'. Under this flow, each point of the surface is transported outwards along the normal direction in that point, with a speed that is determined by the inverse mean curvature $(1/p)$ in that point. It is possible to prove that the surface always ends up as an asymptotic sphere, for which the Geroch mass becomes the ADM mass (provided that the spatial metric is asymptotically flat) [6]. If the starting surface had a Geroch mass which was greater than or equal to zero, then the ADM mass must be positive; and this fact is what is called the positive energy theorem.[15] [9] [10] [7]

The fact that the Geroch mass requires a choice of foliation of spacetime means that it is not intrinsic to the surface of which it is evaluated (in a given spacetime). As such, the Geroch mass is not very well motivated as a definition of quasi-local mass; nor was it intended to be. It was introduced for another purpose, for which it was very useful, but otherwise suffers from the same insufficiencies as the Hawking mass. Nevertheless, it is of interest to study the Geroch mass, since it has proven to be such a useful tool in related problems.

4.1 Sphere in Minkowski - Hyperbolic slicing

We shall now give some explicit examples of the Geroch mass of spheres in different folitations of the Minkowski spacetime. If the foliation is chosen to be the hypersurfaces of constant $t = \tau$ in regular coordinates, then the Geroch mass coincides with the Hawking mass, which is zero. We shall therefore look at some more interesting foliations, for which the two masses do not coincide. The calculations will be very similar to the above calculations of the Hawking mass, so we will leave out steps that are similar.

Let $\mathcal{M}$ be the Minkowski spacetime, described in spherical polar coordinates by the metric

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{85}$$

To begin with, we shall define a foliation of spacetime. Let Σ_τ be the hyperbolic level hypersurfaces of the function

$$\phi(x) = t^2 - r^2 \tag{86}$$

[14] Under some conditions, detailed in the next section.

[15] It is worth nothing that Geroch's proof was not complete. In some situations, the mean curvature can go to zero. The surface will then make a sudden 'jump' outwards, and it was not clear at the time whether the monotonicity applied in those cases. Huisken and Ilmanen later on proved that the Geroch mass was indeed monotone under these circumstances, as part of their proof of the Riemannian Penrose inequality.

Such that the hypersurfaces Σ_τ are surfaces of constant $\phi = \tau^2$. These surfaces may be parameterized as

$$t = \tau \cosh \sigma \quad \text{and} \quad r = \tau \sinh \sigma \tag{87}$$

so that they are coordinatized by $y^a = (\sigma, \theta, \varphi)$ at any constant value of τ (notice the use of the index a to signify that this is not a vector on spacetime). These hypersurfaces are now our idea of *space*, as in all points of constant time τ. We may now, if we like, completely forget about spacetime and do all our work within Σ_τ, treating τ more like a time parameter in the sense of Newtonian mechanics.

Using the parameterization, one finds that the induced metric on this hypersurface is

$$\gamma_{ab} = \tau^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sinh^2 \sigma & 0 \\ 0 & 0 & \sinh^2 \sigma \sin^2 \theta \end{pmatrix} \tag{88}$$

Next, we would like to embed a sphere into these hypersurfaces. A sphere $\mathcal{S}_\tau$ is given by constant coordinate $r = R$, which means we must have $\tau \sinh \sigma = R$, for constant τ. We can write this as a level surface to the function

$$\psi = \tau \sinh \sigma \tag{89}$$

and we may parameterize the surface in terms of coordinates on Σ_τ as

$$\sigma = \operatorname{arcsinh} \frac{R}{\tau}, \quad \theta = u, \quad \varphi = v \tag{90}$$

The next step is to find the spacelike normal vector field of $\mathcal{S}_\tau$. This vector must be tangential to the hypersurface Σ_τ, since the surface is contained within it. Therefore, it is beneficial to solve for the normal vector while working in coordinates $y^a = (\sigma, \theta, \varphi)$ on the hypersurface. We get

$$n_a = N \partial_a F_2 = \begin{pmatrix} \frac{1}{\tau}, & 0, & 0 \end{pmatrix} \tag{91}$$

which has been normalized with respect to the hypersurface metric $(\gamma_\Sigma)_{ab}$. Now, while we may in principle continue working entirely within the hypersurface, it is in fact quicker to go back to spacetime (this saves us from having to calculate the Christoffel symbols on Σ_τ). To do this, we must calculate the 'push-forward' of n_a:

$$n^\mu = n^a \frac{\partial x^\mu}{\partial y^a} = \frac{1}{\tau} \begin{pmatrix} \frac{\partial t}{\partial \sigma}, & \frac{\partial r}{\partial \sigma}, & \frac{\partial \theta}{\partial \sigma}, & \frac{\partial \varphi}{\partial \sigma} \end{pmatrix} = \begin{pmatrix} \sinh \sigma, & \cosh \sigma, & 0, & 0 \end{pmatrix} = \frac{1}{\tau} \begin{pmatrix} r, & t, & 0, & 0 \end{pmatrix} \tag{92}$$

The push-forward may look like a reversed projection, but it is essentially only a re-expression of the same vector in different coordinates.[16] Thus, by construction, this vector is tangential to Σ_τ.

We may now calculate the second fundamental form of $\mathcal{S}_\tau$ with respect to n_μ in ordinary fashion. Using the natural parameterization $\theta = u$ and $\varphi = v$ to coordinatize the surface, one finds that

$$K_{ij}(\vec{n}) = -n_\mu \underbrace{\frac{\partial^2 x^\mu}{\partial u^i \partial u^j}}_{=0} - n_\mu \Gamma^\mu_{\alpha\beta} \frac{\partial x^\alpha}{\partial u^i} \frac{\partial x^\beta}{\partial u^j} = -\frac{t}{\tau} \Gamma^r_{\alpha\beta} \frac{\partial x^\alpha}{\partial u^i} \frac{\partial x^\beta}{\partial u^j} \tag{93}$$

[16] So that if one projects an arbitrary spacetime vector onto the surface, and then calculates its push-forward, one does not, in general, recover the original vector.

(since $\Gamma^t_{\alpha\beta} = 0$ in Minkowski). Inserting the remaining Christoffel symbols leads to

$$K_{ij}(\vec{n}) = \tau \sinh\sigma \cosh\sigma \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 u \end{pmatrix} \tag{94}$$

The metric on $\mathcal{S}_\tau$ is given by setting $d\sigma = 0$ in the metric on Σ_τ.

$$\gamma_{ij} = \tau^2 \begin{pmatrix} \sinh^2\sigma & 0 \\ 0 & \sinh^2\sigma \sin^2 u \end{pmatrix} = R^2 \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 u \end{pmatrix} \tag{95}$$

Yielding the mean curvature:

$$p = \gamma^{ij} K_{ij}(\vec{n}) = \frac{2}{\tau \tanh\sigma} \tag{96}$$

Using the area element $d\mathcal{S} = \sqrt{\gamma} d^2u = \tau^2 \sinh^2\sigma \sin u du dv$, we may now calculate the Geroch mass:

$$\begin{aligned} M_G &= \sqrt{\frac{A}{16\pi}}\left(1 - \frac{1}{16\pi}\oint_{\mathcal{S}} p^2 d\mathcal{S}\right) = \sqrt{\frac{A}{16\pi}}\left(1 - \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \frac{\tau^2 \sinh^2\sigma \sin u du dv}{\tau^2 \tanh^2\sigma}\right) \\ &= \sqrt{\frac{A}{16\pi}} \underbrace{\left(1 - \cosh^2\sigma\right)}_{-\sinh^2\sigma} = -\sqrt{\frac{4\pi\tau^2\sinh^2\sigma}{16\pi}} \sinh^2\sigma = -\frac{\tau \sinh^3\sigma}{2} = -\frac{R^3}{2\tau^2} \end{aligned} \tag{97}$$

Since the area of a sphere is $A = 4\pi R^2 = 4\pi\tau^2 \sinh^2\sigma$.

We see that the Geroch mass is strictly negative. It is worth noting that it decreases monotonically with the radius of the sphere, which is due to the nature of the hypersurface that we have selected. As will be demonstrated in section 5, the Geroch mass is only monotonically increasing with a maximal hypersurface, which means that their mean extrinsic curvature is zero; and that is not the case here. As remarked upon earlier, the hypersurface becomes a light cone at $\tau = 0$; in this case the Geroch mass diverges to negative infinity. The same thing happens when the radius goes to very large values, where the hypersurface becomes more and more like a light cone (τ becomes small in comparison to r). At very large τ and very small radii R, the hypersurface is approximately flat, and here the Geroch mass is almost zero.

4.2 Sphere in Schwarzschild - Trumpet slicing

We will now consider an example of the Geroch mass in the Schwarzschild spacetime, using a coordinate description known as the trumpet coordinates [17].

In numerical simulations, it is troublesome to deal with singularities. If one is given the spacetime metric on an initial Cauchy hypersurface and wants to numerically evolve this solution forward in time, singularities will make life hard, since various quantities become infinite. A way around this problem is to work with a foliation of spacetime in which the singularity is avoided. This is the idea behind the trumpet coordinates of Schwarzschild; they define hypersurfaces of constant time that are singularity-free.[17]

These coordinates can be considered to be a generalization of the Painlevé-Gullstrand (PG) coordinates [18]. The PG coordinates are defined from the point of view of an observer falling

[17] However, it is noteworthy that this specific foliation turns out to not be very useful for computer simulations in practice, but it demonstrates the basic idea of avoiding the singularity.

into a Schwarzschild black hole, so that the hypersurfaces of constant time constitute the rest space of this observer. The PG coordinates are often used to demonstrate that there is nothing funny going on at the event horizion of a black hole, and that an observer may pass it without noticing anything special. However, the singularity at $r = 0$ is problematic, both for the infalling observer and the computer simulation.

In regular Schwarzschild coordinates, the radial coordinate r is not actually defined as the distance from the center, since the spacetime does not have a point which can be regarded as its center. Instead, it is indirectly defined via the area of spheres: if a sphere has area $4\pi r^2$, then its radius is r. The idea of the trumpet coordinates is to exclude a bit of the interior of the black hole, and let a sphere with some non-zero area $4\pi R_0^2$ define the points $r = 0$, so that surfaces of constant r have area $4\pi(r + R_0)^2$. In the limit $R_0 \to 0$, one recovers the Painlevé-Gullstrand coordinates, which include the singularity.

The name "trumpet coordinates" refers to the fact that the spheres of constant r gets smaller and smaller as $r \to 0$, but eventually reach a minimum size. In that sense, the geometry of the spatial slices of constant time is somewhat similar to a trumpet.

In the trumpet coordinates, the Schwarzschild spacetime is given by the line element

$$ds^2 = -f dt^2 + \frac{2f_1}{r} dt dr + f_2^2 (dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2) \tag{98}$$

where

$$f = \left(1 - \frac{2M}{r + R_0}\right), \quad f_1 = \sqrt{2r(M - R_0) + R_0(2M - R_0)}, \quad f_2 = 1 + \frac{R_0}{r}, \quad (0 < R_0 \leq M) \tag{99}$$

and R_0 is a parameter defining the area of the innermost sphere, which must be chosen. As mentioned, the PG coordinates are recovered in the case when $R_0 \to 0$. The case $R_0 = M$ has a special status: it gives a metric which approaches Minkowski as $1/r$ when r is very large; that is to say, the deviation from Minkowski falls off as $1/r$. This classifies it as an asymptotically flat metric. This, in turn, means that the hypersurfaces of constant time have an asymptotically flat metric, and it is in this case that the Geroch mass approaches the ADM mass of the spacetime for asymptotically large spheres [6].

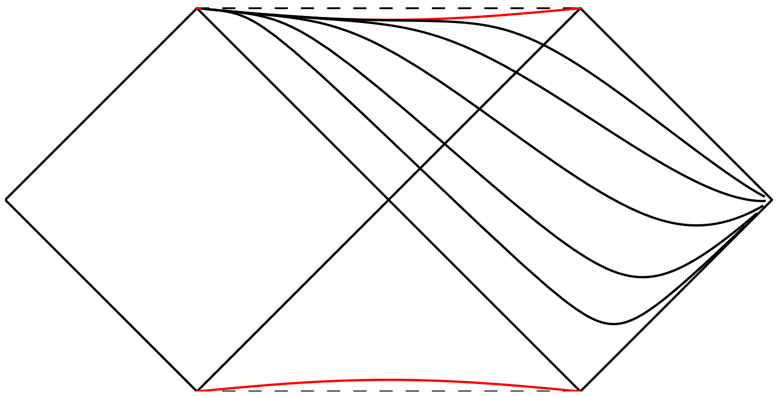

Figure 6: Penrose diagram illustrating hypersurfaces of constant $t = \tau$ in the trumpet coordinates, plotted here for $R_0 = 0.995M$,[19] and $\tau = -10, -7.5, -5, -2.5, 0$. Notice how the hypersurfaces avoids the singularity (which is the dashed line at the top of the diagram) by asymptoting to the curved surface below it, corresponding to the inner sphere at R_0.

These coordinates are constructed in such a way that the spatial metric at any constant time is conformally flat, which makes the hypersurfaces of constant time easy to deal with. We are going to use these hypersurfaces to foliate spacetime and then calculate the Geroch mass of embedded spheres. In other words, let the hypersurfaces Σ_τ be surfaces of constant $t = \tau$, illustrated in the Penrose diagram in figure 6.

These hypersurfaces are then coordinatized by $y^a = (r, \theta, \varphi)$. It is easy to see they have a conformally flat metric, as per construction:

$$(\gamma_\Sigma)_{ab} = f_2^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \tag{100}$$

Next, we define the sphere $\mathcal{S}_\tau$ as the sphere of constant $r = R$, embedded within Σ_τ. This has the spacelike normal

$$n_a = \begin{pmatrix} f_2, & 0, & 0 \end{pmatrix} \quad \text{and} \quad n^a = \begin{pmatrix} \frac{1}{f_2}, & 0, & 0 \end{pmatrix} \tag{101}$$

and its push-forward to $\mathcal{M}$ is

$$n^\mu = n^a \frac{\partial x^\mu}{\partial y^a} = \frac{1}{f_2} \begin{pmatrix} \frac{\partial t}{\partial r}, & \frac{\partial r}{\partial r}, & \frac{\partial \theta}{\partial r}, & \frac{\partial \varphi}{\partial r} \end{pmatrix} = \begin{pmatrix} 0, & \frac{1}{f_2}, & 0, & 0 \end{pmatrix} \tag{102}$$

with covariant counterpart

$$n_\mu = n^\nu g_{\nu\mu} = \begin{pmatrix} \frac{f_1}{r f_2}, & f_2, & 0, & 0 \end{pmatrix} \tag{103}$$

Now, we choose the natural parameterization $\theta = u$ and $\varphi = v$, which yields the induced metric

$$\gamma_{ij} = f_2^2 R^2 \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 u \end{pmatrix} \tag{104}$$

The second fundamental form is then

$$K_{ij}(\vec{n}) = -n_\mu \underbrace{\frac{\partial^2 x^\mu}{\partial u^i \partial u^j}}_{=0} - n_\mu \Gamma^\mu_{\alpha\beta} \frac{\partial x^\alpha}{\partial u^i} \frac{\partial x^\beta}{\partial u^j} = -\Big(\frac{f_1}{R f_2} \Gamma^t_{\alpha\beta} + f_2 \Gamma^r_{\alpha\beta} \Big) \frac{\partial x^\alpha}{\partial u^i} \frac{\partial x^\beta}{\partial u^j} = R \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix} \tag{105}$$

which requires the Christoffel symbols of the Schwarzschild spacetime in trumpet coordinates; these are given explicitly in appendix A since these coordinates are not in common usage. This gives the mean curvature

$$p = \gamma^{ij} K_{ij}(\vec{n}) = \frac{1}{f_2^2 R^2} \cdot R + \frac{1}{f_2^2 R^2 \sin^2\theta} \cdot R \sin^2\theta = \frac{2}{R f_2^2} = \frac{2}{R} \left(1 + \frac{R_0}{R} \right)^{-2} = \frac{2R}{(R + R_0)^2} \tag{106}$$

So that the Geroch mass is

$$\begin{aligned} M_G &= \sqrt{\frac{A}{16\pi}} \Bigg(1 - \frac{1}{16\pi} \oint_{\mathcal{S}} p^2 d\mathcal{S} \Bigg) = \sqrt{\frac{A}{16\pi}} \Bigg(1 - \frac{R^2}{(R + R_0)^4} \frac{1}{4\pi} \underbrace{\oint_{\mathcal{S}} d\mathcal{S}}_{4\pi(R+R_0)^2} \Bigg) \\ &= \frac{R + R_0}{2} \left(1 - \frac{R^2}{(R + R_0)^2} \right) = R_0 - \frac{R_0^2}{2(R_0 + R)} \end{aligned} \tag{107}$$

[19]The case $R_0 = M$ is perhaps most interesting, but it is numerically problematic to plot. Also, the inner surface $r = 0$ becomes hard to see if R_0 is not close to M, which is why this value for R_0 was chosen.

It is easy to see that this expression is positive for any choice of R and R_0, and that it increases monotonely from $M_G = R_0/2$ at $R = 0$ to $M_G = R_0$ as $R \to \infty$. The Geroch mass can thus not exceed the mass of the spacetime, since $R_0 \leq M$. Setting $R_0 = 0$ gives the Painlevé-Gullstrand coordinates, and we note that the Geroch mass is identically zero in this case. Setting $R_0 = M$ gives the asymptotically flat metric, and we note that in this case the Geroch mass goes to $M_G = M$ as $R \to \infty$, which is the ADM mass of the spacetime. The two masses agree only in this case, which is because any other choice of R_0 yields a metric which falls off to Minkowski too slow to classify as asymptotically flat; so that the Geroch mass does not approach the ADM mass (while the ADM mass is defined for any choice of R_0, since it is intrinsic to the spacetime itself).

5 Inverse mean curvature flow

While the Geroch mass is problematic as a definition of mass, Geroch showed that his definition of mass is monotonely increasing under the inverse mean curvature flow (IMCF), which is a kind of geometric flow. In this section, we shall perform a thorough analysis of the inverse mean curvature flow and the monotonicity of the Geroch mass under this flow.

The idea behind a geometric flow is to let a surface evolve continuously outwards (or inwards), so that one generates a family of non-intersecting surfaces that contain eachother. A geometric flow may be defined by the equation

$$\frac{\partial x^a}{\partial \sigma} = v(x) n^a(x) \tag{108}$$

where x^a is the parameterization of the surface, n^a is the normal vector field of the surface and σ is a parameter that drives the evolution. The local speed of the flow (the speed at which the surface moves outward at a point on the surface) is determined by $v(x)$. By letting $v(x)$ be a function of the local curvature, $p(x)$, one gets a curvature-dependent flow, which has a lot of applications [19]. A simple physical example that can be modeled as a curvature-dependent flow is an inflating balloon. If the balloon is not round, but has pointy protrusions (where curvature is large), these will tend to stretch and flatten out as the balloon expands. One can account for this by choosing $v(x)$ such that the flow speed is low where curvature is high.

Other applications of geometric flow equations range from modeling the spread of forest fires or the growth of ice crystals in an undercooled liquid, to teaching computers to recognize hand-written text [19]. Most of these applications regard the evolution of some sort of physical boundary surface, unlike the more abstract surfaces that we treat here.

The inverse mean curvature flow is of course given by choosing $v(x)$ to be the inverse of the curvature. The equation then takes the form

$$\frac{\partial x^a}{\partial \sigma} = \frac{1}{p} n^a \tag{109}$$

where it's to be understood that both p and n are functions of position on the surface. This equation may be better understood if we first introduce coordinates on the surface such that $x^a = x^a(u, v)$. Then, the equation demands that these functions $x^a(u, v)$ also depends on a parameter σ; which we may then think of as a third coordinate. The functions $x^a = x^a(u, v; \sigma)$ then describe one two-dimensional surface with internal coordinates u and v for each σ (so (u, v, σ) is a coordinatization of three-dimensional space).

This equation is notoriously difficult to solve. The main reason for this is the very complicated form of the right-hand side; p and n^a are in fact non-linear functions of derivatives of the functions $x^\mu(u, v)$. To solve the equation, one has to insert the general expressions for both these quantities, which yields a complicated non-linear partial differential equation. One then solves the equation and supplies the surface that one wants to evolve as initial condition, in principle.

There are a few highly symmetric cases that can be solved analytically, such as the evolution of a sphere. The mean curvature of a sphere is constant, and, choosing polar coordinates, so is the normal vector. The symmetry of the situation means that the sphere must stay spherical during the flow, so one does not have to solve the equation in general but may instead solve

$$\frac{\partial r}{\partial \sigma} = \frac{r}{2} \tag{110}$$

since the curvature is $p = 2/r$ and the normal vector is entirely in the radial direction. This equation has the solution

$$r(\sigma) = r_0 e^{\sigma/2} \tag{111}$$

where r_0 is the radius of the initial surface. Note that the equation could be written on this form because the shape of the surface does not change, so that the functional form of the mean curvature and normal vector remains the same. A more general surface will change its shape as it evolves, which means that both p and n^a changes. If one were to calculate the mean curvature of a spheroid, say, and insert it as p in the equation above, one would quickly run into problems as the spheroid evolves into something which is no longer a spheroid, which means that the expression for p is invalid and the equation breaks down.

The sphere serves as a good guide to how the inverse mean curvature flow works. One can loosely think of a more general closed surface as a sphere with funny surface features. Any feature which protrudes from the surface will have higher mean curvature than the more spherical parts, which means that it will move outwards slower. The rest of the surface, moving faster, then 'catches up' with the protrusion, so that it flattens out. The converse happens for flatter parts; they will shoot out at higher speed than the rest of the surface, so that they become less flat. Thus, deviations from sphericity become gradually smaller until the surface eventually becomes a sphere.[20]

5.1 Monotonicity of the Geroch mass

As mentioned previously, Geroch showed that the Geroch mass is monotonely increasing under the mean curvature flow [6]. This flow defines a one-parametric family of surfaces, each larger than and containing the previous one, and the statement is that the Geroch mass contained in each surface grows as one moves outward.

Geroch originally introduced the Geroch mass for the sole purpose of proving the positive-energy theorem. It states that the total energy of an asymptotically flat spacetime is positive, provided that the dominant energy condition holds[21] The idea of the proof is rather elegant: for a large, asymptotically spherical surface, the Geroch mass approaches the ADM mass, which is the usual definition of total mass for a spacetime. An arbitrary surface evolving under inverse mean curvature flow will asymptote to a sphere, and the Geroch mass increases as it does so. This means that as long as one can find *any* surface with $M_G \geq 0$, the ADM mass will be positive. This is always possible, since $M_G \to 0^+$ for small spheres about a point. The monotonicity only holds in a maximal hypersurface, but these can always be found.

There are some complications that can arise, however. For example, if one starts with a torus, the torus will grow 'fatter' with the flow, until, at some point, the 'hole' in the middle will vanish. When this happens, the surface goes from being toroidal into something that resembles a spheroid. The topology of the surface makes a very sudden change, which interferes with the smooth flow: the surface makes a 'jump'. In addition, a surface can have flat regions, where the curvature is zero, which also causes jumps in the flow. It is not at all obvious that the Geroch mass is monotone if these things are allowed to happen, and Geroch did not adress this. Therefore, his proof of the positive energy theorem was not complete. However, Huisken and Ilmanen, in their proof of the Riemannian Penrose inequality [7], put the final pieces into

[20] There are some visual examples of this in the below section on a numerical solution of the inverse mean curvature flow.

[21] There are a few different energy conditions that may be assumed in general relativity, and they essentially place restrictions on the matter contents of spacetime, for example by excluding negative matter densities. See for example (ref).

place and proved that the Geroch mass is monotone even during these jumps. Their set-up even required starting the flow with a minimal surface; one which has $p = 0$ everywhere. In what follows, we will not go into any greater detail on these issues, but we shall state the proof of monotonicity as Geroch originally stated it.

Let $\mathcal{S}$ be a two-surface with parameterization $x^\mu = x^\mu(u, v)$ in terms of the surface coordinates $u^i = (u, v)$. Let the surface be embedded within a spatial hypersurface such that its normal vector field n^μ is uniquely defined. Let this surface be the initial condition to the equation

$$\frac{\partial x^\mu}{\partial \sigma} = \phi n^\mu \tag{112}$$

where σ is a parameter that drives the evolution. This defines a family of surfaces $\mathcal{S}_\sigma$. Given that $\phi > 0$ and n^μ is the outwards facing normal, $\mathcal{S}_{\sigma_2}$ contains all of $\mathcal{S}_{\sigma_1}$ if $\sigma_2 \geq \sigma_1$. If ϕ is chosen to be specifically

$$\phi = \frac{1}{p} \tag{113}$$

where p is the mean extrinsic curvature of the surface $\mathcal{S}_\sigma$, then the Geroch mass

$$M_G = \sqrt{\frac{A}{16\pi}} \left(1 - \frac{1}{16\pi} \oint_{\mathcal{S}} p^2 d\mathcal{S} \right) \tag{114}$$

is monotonely increasing along the flow. In other words, its Lie derivative[22] with respect to the flow is positive:

$$\mathcal{L}_{\phi\vec{n}} M_G \geq 0 \tag{115}$$

We will now present the proof of the monotonicity of the Geroch mass as it was originally presented by Geroch in (ref), but in more detail. For this purpose, some theorems from differential geometry will be required, which are derived in appendix B.

Using the identity $\oint R_{\mathcal{S}} d\mathcal{S} = 16\pi$, which is true for any closed surface, the Geroch mass may be written

$$M_G = \sqrt{\frac{A}{16\pi}} \frac{1}{16\pi} \underbrace{\oint_{\mathcal{S}} \left(2R_S - p^2 \right) d\mathcal{S}}_{=W} \tag{116}$$

To prove that the Geroch mass is monotone under the inverse mean curvature flow, one must show that

$$\mathcal{L}_{\frac{1}{p}\vec{n}} M_G \geq 0 \tag{117}$$

To make things easier to read, we shall use the overdot notation ($\dot{M}_G$) to denote the Lie derivative with respect to this flow. We see that

$$\dot{M}_G = \frac{1}{\sqrt{16\pi}} \frac{1}{16\pi} \left(\frac{\dot{A}}{2\sqrt{A}} W + \sqrt{A} \dot{W} \right) \tag{118}$$

Now, recall from section 2.2.4 the formula

$$p = \frac{1}{2} \gamma^{ij} \mathcal{L}_{\vec{n}} \gamma_{ij} = \frac{1}{\sqrt{\gamma}} \mathcal{L}_{\vec{n}} \sqrt{\gamma} \tag{119}$$

[22]For a scalar, the Lie derivative is the usual directional derivative: $\mathcal{L}_{\phi\vec{n}} = \phi n^\mu \nabla_\mu$.

The square root of the determinant of the metric is the area element on the surface, so we can conclude that

$$\frac{1}{p}\mathcal{L}_{\vec{n}}d\mathcal{S} \equiv \mathcal{L}_{\frac{1}{p}\vec{n}}d\mathcal{S} = d\mathcal{S} \tag{120}$$

Since the total area is the integral of the area element, we find (using linearity of the integral) that $\dot{A} = A$. Note how this fact relies on choosing the speed of to flow to be specifically $1/p$. Using this gives

$$\dot{M}_G = \sqrt{\frac{A}{16\pi}}\frac{1}{16\pi}\left(\frac{W}{2} + \dot{W}\right) \tag{121}$$

Moving on,

$$\dot{W} = -\oint 2p\dot{p}d\mathcal{S} - \oint p^2\dot{d\mathcal{S}} = -\oint (2p\dot{p} + p^2)d\mathcal{S} \tag{122}$$

We now require the identity describing the second variation of the area. This identity is famously difficult to derive, but we have supplied a condensed version of the derivation in appendix B. Adapted to the case at hand, it states

$$\dot{p} = -D^iD_i\frac{1}{p} - \frac{1}{2p}\left(p^2 + K_{ij}K^{ij} - R_\Sigma - R_\mathcal{S}\right) \tag{123}$$

where D_i is the covariant derivative on $\mathcal{S}$. This gives

$$\begin{aligned}\dot{W} &= \oint\left(R_\Sigma + K^{ij}K_{ij} + 2pD_aD^a\frac{1}{p} - R_\mathcal{S}\right)d\mathcal{S}\\ &= \oint\left(R_\Sigma + K^{ij}K_{ij} - \frac{1}{2}p^2 + 2pD_aD^a\frac{1}{p}\right)d\mathcal{S} - \underbrace{\oint\left(R_\mathcal{S} - \frac{1}{2}p^2\right)d\mathcal{S}}_{=\frac{1}{2}W}\\ &= \oint\Big(R_\Sigma + \underbrace{(K^{ij} - \frac{1}{2}\gamma^{ij}p)(K_{ij} - \frac{1}{2}\gamma_{ij}p) + 2p^2D_a\frac{1}{p}D^a\frac{1}{p}}_{\equiv U^2}\Big)d\mathcal{S} - \frac{1}{2}W\end{aligned} \tag{124}$$

Where we've factorized some of the terms into squares so that their positivity is evident. Now, Gauss' Theorema Egregium, also found in appendix B, together with Einstein's field equations, allows us to relate the intrinsic and extrinsic curvatures of the hypersurface Σ:

$$R_\Sigma = 2\mu + \kappa_{ij}\kappa^{ij} - \kappa^2 \tag{125}$$

Where κ_{ij} is the second fundamental form of the hypersurface Σ, and κ is the mean extrinsic curvature of Σ (not to be confused with p, which is the mean extrinsic curvature of $\mathcal{S}$). Under the assumption that Σ is a maximal hypersurface, which means that $\kappa^2 = 0$, we find that

$$\dot{M}_G = \sqrt{\frac{A}{16\pi}}\frac{1}{16\pi}\oint(2\mu + \kappa_{ab}\kappa^{ab} + U^2)d\mathcal{S} \tag{126}$$

Both $\kappa_{ab}\kappa^{ab}$ and U^2 are squared quantities, so this integral is strictly positive as long as the energy density μ is not negative, which is a fair assumption[23]. One can thereby conclude that

$$\mathcal{L}_{\frac{1}{p}\vec{n}}M_G \geq 0 \tag{127}$$

as long as the foliation is chosen so that the spatial hypersurfaces are maximal and the dominant energy condition holds.

[23] This is usually called the dominant energy condition (DEC), and essentially means that we've assumed that there is no matter with negative mass. [20]

5.2 A numerical solution of IMCF

Finding solutions to geometric flow equations and understanding them is an important task, not just in the scope of general relativity, but in many other fields as well; as we've seen, these types of equations can be used to model a wide range of problems involving the evolution of boundary surfaces. In many, if not most, cases, it is not possible to solve these equations analytically, so that numerical solutions must be pursued instead. It is therefore important to understand how these kinds of equations may be solved numerically. Part of this thesis was focused on studying the inverse mean curvature flow in a special case and producing a numerical solution, to gain understanding of the problems that needs to be overcome to achieve this.

The inverse mean curvature flow has a simple solution if the surface starts out as a sphere, as has been shown above. Anything more complicated than a sphere lacks such a simple solution, and must be solved numerically in most cases. It was therefore chosen to regard the evolution of nearly spherical surfaces, namely the spheroids that have been studied earlier in this thesis, so that the solution could easily be compared to what was expected for a sphere. It is natural to expect that these surfaces should quickly turn into a sphere under the flow.

Restricting attention specifically to spheroids meant that the algorithm only had to be designed to deal with these, allowing some simplifications to be made. For example, one does not have to take into account non-convex surfaces, which has curvature of both signs - meaning that the inverse curvature diverges where the curvature switches sign. Additionally, since the spheroid is rotationally symmetric, the problem reduces to solving for the evolution of a curve in two dimensions, so that the minimum amount of data points required to represent the solution is small.

The mathematical details of the algorithm will be presented below, while an implementation of the algorihm in MATLAB is presented in appendix C. Figure 7 shows example outputs of the program for an oblate and prolate spheroid, in cross section. As can be seen, the surfaces quickly become spherical, at which point the evolution continues outwards exponentially. While perhaps not so easy to see in the figure, the surfaces tend to 'unfold' to a sphere *first*, in the sense that the surface retains the same overall scale, before it starts growing outwards (exponentially). This can be seen more clearly if the plots are turned into an animated movie, which is included as a function in the MATLAB program.

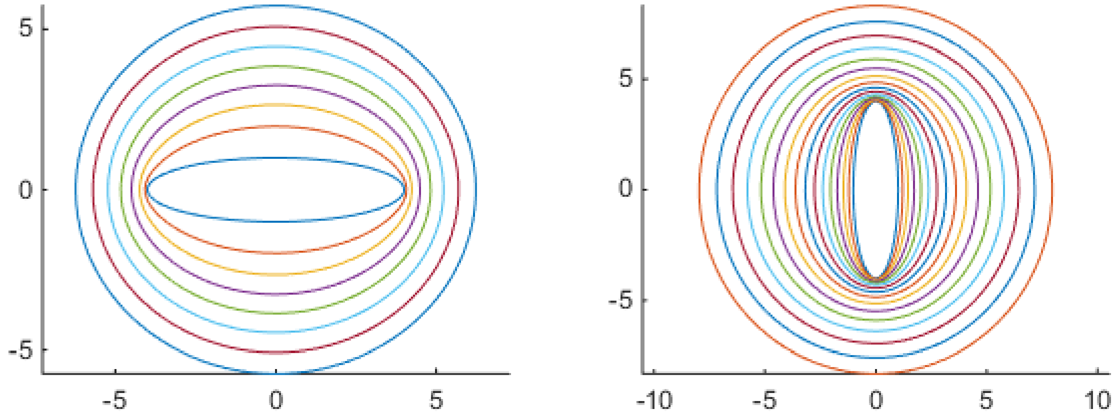

Figure 7: The result of running the algorithm to evolve an oblate spheroid (left), starting out with axes $a = 4$ and $c = 1$, and prolate spheroid (right), with starting axes $a = 1$ and $c = 4$. The oblate spheroid is plotted in 8 steps from $\sigma = 0$ to $\sigma = 1.6$ using $N = 200$ data points, starting from the inner ellipse and evolving outwards in successive steps. The prolate spheroid takes longer to 'unfold' into a sphere, due to the fact that it has much larger curvature overall. It is plotted from $\sigma = 0$ to $\sigma = 3.2$ in 16 steps, also using $N = 200$ data points.

5.2.1 Stating the equations

Previous work on similar equations have shown that solutions based on the parametric formulation, in which the IMCF equation is stated, face significant difficulties [21] [22]. To clarify, the equation reads

$$\frac{\partial x^a}{\partial \sigma} = \frac{1}{p} n^a, \tag{128}$$

where $x^a = x^a(u_1, u_2, \sigma)$ is a parameterization of the surface in terms of the coordinates $u^i = (u_1, u_2)$ on the surface and the evolution parameter σ. To discretize the equation, one would have to express both p and n^a in terms of the parameterization, which is possible but very complicated. Instead, as described by J.A. Sethian in [21], [22] and [19], it is much more efficient to reformulate the equation using level set methods[24].

The idea behind level set methods is to define the surface as a level set of a scalar function, which shall be named ψ, and regard ψ as the quantity that evolves. This yields an equation for the evolution of ψ which is equivalent to the parametric form described above, in that it reproduces the same surface for $\psi =$ constant. We shall now show how this is done in detail.

Let there be a scalar function $\psi(x^a)$ so that

$$\psi(x^a(u^i; \sigma)) - q = 0 \tag{129}$$

for some constant q. Thus, the surfaces $x^a(u^i; \sigma)$ (at any constant value of σ) are level sets of ψ. The normal vector field of the surface is then given by the normalized gradient of ψ:

$$n_a = \frac{\partial_a \psi}{\sqrt{\partial_b \psi \partial^b \psi}} \tag{130}$$

[24]Level set methods are also used by Huisken and Ilmanen in their proof of the Riemannian Penrose inequality [7].

Furthermore, it is evident that

$$\frac{\partial \psi}{\partial \sigma} = \frac{\partial \psi}{\partial x^a}\frac{\partial x^a}{\partial \sigma} = \partial_a \psi \frac{\partial x^a}{\partial \sigma} \tag{131}$$

Now, taking the IMCF equation and contracting it with $\partial_a \psi$ gives

$$\partial_a \psi \frac{\partial x^a}{\partial \sigma} = \frac{1}{p}\partial_a \psi n^a \iff \frac{\partial \psi}{\partial \sigma} = \frac{1}{p}\frac{\partial_a \psi \partial^a \psi}{\sqrt{\partial_b \psi \partial^b \psi}} = \frac{1}{p}\left|\partial \psi\right| \tag{132}$$

according to the above identities. Thus, ψ must obey the equation

$$\frac{\partial \psi}{\partial \sigma} = \frac{1}{p}\left|\partial \psi\right| \tag{133}$$

which is an easier equation to handle than what we started with, since ψ is the only unknown.

Now, since we are restricting ourselves to convex rotationally symmetric surfaces, the problem simplifies further: we only have to deal with the evolution of a cross section of the surface, which is a curve in two dimensions. In standard polar coordinates, this curve can be parameterized as $r = \rho(\theta, \sigma)$. One can then make the ansatz

$$\psi = r - \rho(\theta, \sigma) \quad \text{so that} \quad \psi = 0 \quad \Longrightarrow \quad r = \rho(\theta, \sigma) \tag{134}$$

Inserting this into equation 133 yields

$$-\frac{\partial \rho}{\partial \sigma} = \frac{1}{p}\left|\partial(r-\rho)\right| = \frac{1}{p}\left|\begin{pmatrix} 1, & -\rho_\theta, & 0 \end{pmatrix}\right| = \frac{1}{p}\sqrt{1+\frac{\rho_\theta^2}{\rho^2}} \tag{135}$$

where we've used the notation $\frac{\partial \rho}{\partial \theta} \equiv \rho_\theta$ for compactness. The equation that needs be solved is then

$$\frac{\partial \rho}{\partial \sigma} + \frac{1}{p}\sqrt{1+\frac{\rho_\theta^2}{\rho^2}} = 0 \tag{136}$$

where p is a (complicated) function of ρ and its derivatives, which we will now solve for.

5.2.2 Mean curvature of a rotationally symmetric closed surface

The assumption so far is that the surface is rotationally symmetric and convex. It is therefore sufficient to solve for the mean curvature of an arbitrary convex rotationally symmetric surface (as opposed to using a more general expression that would work for any surface).

As stated above, an arbitrary convex rotationally symmetric surface can be specified by giving its radius as a function of the polar angle: $r = \rho(\theta)$. The metric on the surface is then given by substituting $dr = \frac{\partial \rho}{\partial \theta} d\theta \equiv \rho_\theta d\theta$ into the Euclidean metric $ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2$, giving

$$ds^2 = (\rho^2 + \rho_\theta^2)d\theta^2 + \rho^2 \sin^2\theta d\varphi^2 \quad \Longrightarrow \quad \gamma_{ij} = \begin{pmatrix} \rho^2 + \rho_\theta^2 & 0 \\ 0 & \rho^2 \sin^2\theta \end{pmatrix} \tag{137}$$

The surface is a level surface of the function $\psi = r - \rho$, which gives the normal vector

$$n_a = N\partial_a \psi = N\begin{pmatrix} 1, & -\rho_\theta, & 0 \end{pmatrix}, \quad n_a n^a = 1 \quad \Longrightarrow \quad N = \left(1 + \frac{\rho_\theta^2}{\rho^2}\right)^{-1/2} \tag{138}$$

The second fundamental form is given by

$$K_{ij}(\vec{n}) = -n_a \frac{\partial x^a}{\partial u^i \partial u^j} - n_a \Gamma^a{}_{bc} \frac{\partial x^a}{\partial u^i} \frac{\partial x^b}{\partial u^j} \tag{139}$$

Thus

$$K_{\theta\theta}(\vec{n}) = -n_a \frac{\partial^2 x^a}{\partial \theta^2} - n_a \Gamma^a{}_{bc} \frac{\partial x^a}{\partial \theta} \frac{\partial x^b}{\partial \theta} = -N\rho_{\theta\theta} - n_a \left(\Gamma^a{}_{\theta\theta} + \Gamma^a{}_{rr}\rho_\theta^2 + 2\Gamma^a{}_{r\theta}\rho_\theta\right) \tag{140}$$

$$= -N\rho_{\theta\theta} - N\Gamma^r{}_{\theta\theta} + 2N\rho_\theta^2 \Gamma^\theta{}_{r\theta} = N\left(\rho - \rho_{\theta\theta} + 2\frac{\rho_\theta^2}{\rho}\right) \tag{141}$$

$$K_{\varphi\varphi}(\vec{n}) = -n_a \frac{\partial^2 x^a}{\partial \varphi^2} - n_a \Gamma^a{}_{bc} \frac{\partial x^a}{\partial \varphi} \frac{\partial x^b}{\partial \varphi} = -n_a \Gamma^a{}_{\varphi\varphi} = -N(\Gamma^r{}_{\varphi\varphi} - \rho_\theta \Gamma^\theta{}_{\varphi\varphi}) \tag{142}$$

$$= N(\rho \sin^2\theta - \rho_\theta \sin\theta\cos\theta) \tag{143}$$

We do not need to calculate the off-diagonal elements since the next step is to take the trace with respect to a diagonal metric. We get the mean curvature:

$$p = \gamma^{ij} K_{ij}(\vec{n}) = \frac{1}{\rho^2 + \rho_\theta^2} N\left(\rho - \rho_{\theta\theta} + 2\frac{\rho_\theta^2}{\rho}\right) + \frac{1}{\rho^2 \sin^2\theta} N(\rho\sin^2\theta - \rho_\theta \sin\theta\cos\theta) \tag{144}$$

$$= \frac{1}{\rho}\left(1 + \frac{\rho_\theta^2}{\rho^2}\right)^{-1/2} \left(\frac{\rho^2 - \rho\rho_{\theta\theta} + 2\rho_\theta^2}{\rho^2 + \rho_\theta^2} + 1 - \frac{\rho_\theta}{\rho}\cot\theta\right) \tag{145}$$

By simply disregarding the φ-direction (leaving only the $\theta\theta$-component of the metric and second fundamental form), one sees that the *contour* $r = \rho(\theta)$ has the curvature

$$p_{curve} = \frac{1}{\rho}\left(1 + \frac{\rho_\theta^2}{\rho^2}\right)^{-1/2} \left(\frac{\rho^2 - \rho\rho_{\theta\theta} + 2\rho_\theta^2}{\rho^2 + \rho_\theta^2}\right) \tag{146}$$

which is contained in the mean curvature of the surface of revolution. The curvature added by revolving the curve is thus

$$p_\varphi = \frac{1}{\rho}\left(1 + \frac{\rho_\theta^2}{\rho^2}\right)^{-1/2} \left(1 - \frac{\rho_\theta}{\rho}\cot\theta\right) \tag{147}$$

The cotangent in this term is obviously problematic for numerical evaluations, since it diverges at $\theta = 0$ and $\theta = \pi$. However, this is merely an artifact of the coordinate system: the product $\rho_\theta \cot\theta$ will always be finite so long as the surface is continuously differentiable (as this requires ρ_θ to go to zero at the poles faster than $\cot\theta$ diverges), and we shall only be concerned with such surfaces. But a computer cannot evaluate $\rho_\theta \cot\theta$ directly if θ is within working precision of the poles, so one needs to approximate the limits at the poles numerically. This can be done rather easily by simply using the value of p_φ at the closest neighbouring point. The error in this approximation will be of order of the spatial resolution (the distance between each point used to represent the solution) or smaller, which can be seen by Taylor expanding $\rho_\theta \cot\theta$ around the poles.

5.2.3 Discretizing

The differential equation may be discretized using a FTCS (Forward-Time Central-Space) scheme[25], which means that the 'time'[26] derivative $\frac{\partial\rho}{\partial\sigma}$ is discretized as a 'forward difference' and the spatial derivative $\frac{\partial\rho}{\partial\theta}$ as a 'central difference', defined as follows

$$\frac{\partial\rho(\theta,\sigma)}{\partial\sigma} = \lim_{\delta\sigma\to 0}\frac{\rho(\theta,\sigma+\delta\sigma)-\rho(\theta,\sigma)}{\delta\sigma} \quad \underset{\to}{\text{Discretized as}} \quad \left(\frac{\partial\rho}{\partial\sigma}\right)_n^i = \frac{\rho_n^{i+1}-\rho_n^i}{\Delta\sigma} \tag{148}$$

$$\frac{\partial\rho(\theta,\sigma)}{\partial\theta} = \lim_{\delta\theta\to 0}\frac{\rho(\theta+\delta\theta,\sigma)-\rho(\theta,\sigma)}{\delta\theta} \quad \underset{\to}{\text{Discretized as}} \quad \left(\frac{\partial\rho}{\partial\theta}\right)_n^i = \frac{\rho_{n+1}^{i}-\rho_{n-1}^i}{2\Delta\theta} \tag{149}$$

where ρ_n^i is the discrete approximation of $\rho(\theta,\sigma)$:

$$\rho_n^i = \rho(\theta_n,\sigma_i) \quad \text{and} \quad \theta_n = n\cdot\Delta\theta, \quad \sigma_i = i\cdot\Delta\sigma \tag{150}$$

where $\Delta\theta$ is determined by the spatial resolution, i.e. the number of data points used to specify the initial surface: $\Delta\theta = \pi/N$, where N is the number of points. The 'time' step $\Delta\sigma$ is determined by a stability condition, which will be detailed in the next subsection. Differentiating again, the second spatial derivative becomes

$$\left(\frac{\partial^2\rho}{\partial\theta^2}\right)_n^i = \frac{\left(\frac{\partial\rho}{\partial\theta}\right)_{n+1}^i - \left(\frac{\partial\rho}{\partial\theta}\right)_{n-1}^i}{2\Delta\theta} = \frac{\rho_{n+2}^i - 2\rho_n^i + \rho_{n-2}^i}{4\Delta\theta^2} \tag{151}$$

In terms of the discretized time-derivative, the differential equation reads

$$\frac{\rho_n^{i+1}-\rho_n^i}{\Delta\sigma} = -\frac{1}{p}\sqrt{1+\frac{\rho_\theta^2}{\rho^2}} \equiv -F(\rho,\rho_\theta,\rho_{\theta\theta}) \tag{152}$$

therefore, given the solution at some time i, one may propagate it forward in time using

$$\rho_n^{i+1} = \rho_n^i - F(\rho,\rho_\theta,\rho_{\theta\theta})\cdot\Delta\sigma \tag{153}$$

where the spatial derivatives of the right hand side are taken to be the discretized derivatives detailed above.

[25]Good references for numerical algorithms are for example [23] and [24].

[26]We shall continue referring to σ as a time-parameter, since it is easiest to think of it in this way. However, recall that it was introduced simply as a parameter for a family of surfaces.

5.2.4 Stability

All computer algorithms suffer from numerical errors, mainly due to the fact that a computer cannot work with infinite precision, which means that it rounds all numbers up to a significant digit determined by the working precision. There are other sources of errors too, such as approximations that have been built into the algorithm. These errors will propagate forward through each iteration, and may grow larger each time; in addition to the fact that new round-off errors are added in each individual numerical evaluation.

These errors may in fact grow so fast that they overtake the solution in a very short time, rendering the computation useless. How fast the errors grow will depend a lot on the form of the differential equation, and the one we're concerned with is especially sensitive. From a qualitative perspective, the reason for this can be seen quite easily: small random errors causes a surface that is supposed to be completely smooth to have irregularities, which means added *curvature*. From a distance, the surface will appear to be smooth, but if one looks *very* closely, it is irregular.

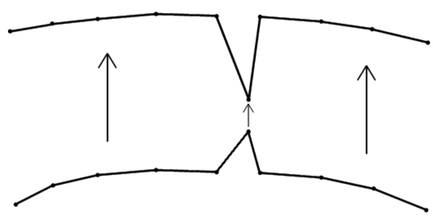

Figure 8: Large spikes in curvature propagate outward slower, which can cause growing errors.

The errors are random, and thus causes local spikes in the curvature. Since the speed of the evolution is determined by the inverse of the curvature, the surface moves much slower than it's supposed to at these spikes - meaning that new curvature is created as the rest of the surface overshoots the irregularity. The remedy for this problem is to evolve the surface in sufficiently small steps for this 'overshooting' not to happen. We will now detail how this is done.

Given the solution at some time step i, it is propagated forward in time by

$$\rho_n^{i+1} = \rho_n^i - F(\rho, \rho_\theta, \rho_{\theta\theta}) \cdot \Delta\sigma \tag{154}$$

however, because of the aforementioned errors, ρ_n^i is not the *true* solution. We can write

$$\rho_n^i = \hat{\rho}_n^i + \epsilon_n^i \tag{155}$$

where $\hat{\rho}_n^i$ is the true solution and ϵ_n^i is the error. We shall now show how the error grows with each iteration. This is given by taking the difference between the propagation of the error-containing ρ_n^i and the true solution $\hat{\rho}_n^i$:

$$\epsilon_n^{i+1} = (\hat{\rho} + \epsilon)_n^{i+1} - \hat{\rho}_n^{i+1} \tag{156}$$

Furthermore, the errors will have a random spatial distribution, and will be of a characteristic size. We can write this as

$$\epsilon_n^i = f_n^i \epsilon^i \tag{157}$$

where ϵ^i is the characteristic size of the errors at time i, and f_n^i is of order unity and describes the random spatial fluctuations of the error. We then have

$$\epsilon_n^{i+1} = (\hat{\rho} + \epsilon)_n^{i+1} - \hat{\rho}_n^i = \epsilon_n^i - \Big(F(\hat{\rho} + \epsilon) - F(\hat{\rho})\Big) \cdot \Delta\sigma \tag{158}$$

The difference $F(\hat{\rho}+\epsilon) - F(\hat{\rho})$ may be evaluated analytically using calculus of variations:

$$F(\hat{\rho} + \epsilon) - F(\hat{\rho}) = \left(\frac{\partial F}{\partial \rho}\epsilon + \frac{\partial F}{\partial \rho_\theta}\epsilon_\theta + \frac{\partial F}{\partial \rho_{\theta\theta}}\epsilon_{\theta\theta} \right) \tag{159}$$

We may discretize this expression as

$$F(\hat{\rho}+\epsilon) - F(\hat{\rho}) \to \left.\frac{\partial F}{\partial \rho}\right|_{\rho_n^i,(\rho_\theta)_n^i,(\rho_{\theta\theta})_n^i} f_n^i \epsilon^i + \left.\frac{\partial F}{\partial \rho_\theta}\right|_{\rho_n^i,(\rho_\theta)_n^i,(\rho_{\theta\theta})_n^i} (f_\theta)_n^i \epsilon^i + \left.\frac{\partial F}{\partial \rho_{\theta\theta}}\right|_{\rho_n^i,(\rho_\theta)_n^i,(\rho_{\theta\theta})_n^i} (f_{\theta\theta})_n^i \epsilon^i \tag{160}$$

where we have inserted the discrete versions of ρ and its derivatives into the analytical derivatives of F, and taken the spatial derivatives of the random error distribution f_n^i. Now, we make the observation that if $\left|f_n^i\right| \leq 1$, then

$$(f_\theta)_n^i \leq \frac{2}{\Delta\theta} \tag{161}$$

Because the difference between two neighbouring points is at most 2, and they are separated by a distance $\Delta\theta$. Equivalently,

$$(f_{\theta\theta})_n^i \leq \frac{4}{\Delta\theta^2} \tag{162}$$

Thus, in the absolute worst case scenario, the last term of equation 160 is much larger than the rest[27], so that the main contribution to the growth of the error comes from this term. We now make the observation that the growth of the error is also proportional to the size of the time step, $\Delta\sigma$:

$$\epsilon_n^{i+1} = \epsilon_n^i - \Big(F(\hat{\rho}+\epsilon) - F(\hat{\rho})\Big) \cdot \Delta\sigma \tag{163}$$

So that by choosing the time step, we can make the growth of the error (per iteration) as small as we want. To get a small error growth, the time step has to be small enough to suppress all three terms in equation 160, and since the third term is the dominant one, we may choose the time-step such that

$$\max_n \left| \left.\frac{\partial F}{\partial \rho_{\theta\theta}}\right|_{\rho_n^i,(\rho_\theta)_n^i,(\rho_{\theta\theta})_n^i} \right| \cdot (f_{\theta\theta})_n^i \cdot \Delta\sigma^i \leq 1 \tag{164}$$

where we have now introduced a time index i on the time-step, since this choice has to be made each iteration. We've also taken the maximum value over the whole surface, to make sure that the time step is small enough to work for the entire surface. Using the worst case approximation for $(f_{\theta\theta})$, we get

$$\Delta\sigma^i \leq \frac{\Delta\theta^2}{4} \left(\max_n \left| \left.\frac{\partial F}{\partial \rho_{\theta\theta}}\right|_{\rho_n^i,(\rho_\theta)_n^i,(\rho_{\theta\theta})_n^i} \right| \right)^{-1} \tag{165}$$

With this choice, we see that[28]

$$\left|\epsilon_n^{i+1} - \epsilon_n^i\right| \leq \epsilon^i \tag{166}$$

Recall that ϵ^i was the characteristic size of the errors at time i. This means that the added error in each step is bounded by the size of the error in the previous step, so that it cannot suddenly diverge and destabilize the algorithm. These errors can in fact grow extremely large in a single or a few iterations if one chooses too large a time step.

An important fact that should be noted is that this does not show that the errors will not continue growing. They may, in fact, grow very fast according to the above analysis. What should be emphasized, however, is that it guarantees that the errors will not suddenly 'blow up' and render the solution useless in a single or a few iterations. In practice, the algorithm runs very well with this choice of time-step, and appears to converge as the spatial resolution

[27] Provided that the derivatives in the other terms are bounded, but this can be verified manually.

[28] Strictly speaking, this is under the assumption that the first two terms of equation 160 are negligible, but this can always be made true by making $\Delta\theta$ sufficiently small.

is increased; so random errors does not seem to affect the result much. This can probably be attributed to the fact that differential equation itself has the property that it tends to smooth out irregularities in the surface. However, this cannot happen if the time-step is too large to resolve the irregularities.

We note that the chosen time-step is proportional to the square of the spatial resolution, which means that a high-fidelity calculation may take a very long time to compute. The program includes an option designed to speed up the calculation significantly, at the loss of accuracy, which is accomplished by smoothing out the first derivative of ρ. This means that the random distribution of errors in the second derivative will not be proportional to $\Delta\theta^2$, but closer to $\Delta\theta$, so that the time step can be chosen proportional to $\Delta\theta$. This can be utilized to make 'fast and loose' simulations, for example if one wishes to experiment with the parameters of the program.

Finally, we remark that, as may be inferred from the proof of the monotonicity of the Geroch mass (section 5.1), the total curvature of the surface as measured by the integral $\int p^2 dS$ should decrease under the flow. Included in the program is the option to plot the change of this measure during the evolution, which may be used as an extra check to see that nothing has gone wrong.

6 Non-zero cosmological constant

6.1 Hawking and Geroch mass in spacetimes with non-zero Λ

So far, all the spacetimes that have been considered have been without a cosmological constant, Λ. The cosmological constant was originally introduced by Einstein so that his field equations would have a static solution, one which does not expand or contract, because it was thought at the time that this must be true for the universe. When Hubble showed that the universe does in fact appear to be expanding, Einstein called the cosmological constant his 'biggest blunder', and it was subsequently assumed that it must be zero.

Current observations [25] indicate that the cosmological constant is non-zero and positive, albeit very small, which makes spacetimes with non-zero Λ interesting to study.[29] Representing a constant energy density, also called dark energy, which contributes a negative pressure throughout the entire universe, the cosmological constant is believed to be responsible for the accelerating expansion of the universe. This may sound counterintuitive: a negative pressure sounds like it would 'suck things in'. The explanation is that regular pressure causes ordinary gravitational attraction, owing to the fact that the matter is more energetic. A negative pressure, on the other hand, does the opposite: it pushes the universe apart.

It is necessary to decide whether or not this dark energy should be thought of as a mass and whether or not it should be included when considering quasi-local mass. There is no direct answer to this question yet. If the cosmological constant really is constant over the entire universe, and if there is no interaction or interchange between dark energy and regular energy, then one can safely disregard its contribution to quasi-local mass. If one does choose to include the dark energy, then the problem that asymptotically large spheres does not have a finite mass arises.

It is worth noting that the current understanding of the nature of the cosmological constant is very limited. Predictions from quantum field theory based on the energy of vacuum disagree with the measured value by so many orders of magnitude that it has been called the "worst theoretical prediction in the history of physics" [26]. This is commonly known as the cosmological constant problem.

In this section, we will show how the Hawking and Geroch masses are affected by the addition of a non-zero cosmological constant, and how these can be modified to exclude contributions from it, in some simple cases. These corrections have been suggested previously by [27], but we will also show how they affect the monotonicty of the Geroch mass.

6.2 Hawking mass in anti-de Sitter

We will begin by calculating the Hawking mass of a sphere in the anti-de Sitter (AdS) spacetime, which is a vacuum solution of the Einstein field equations that exhibits a negative cosmological constant. As it stands, the curvature caused by the non-zero cosmological constant should contribute to the Hawking mass. Knowing this contribution will allow us to modify the Hawking mass so that the resulting mass is zero, which would be expected of a sphere in vacuum.

[29] Including spacetimes with negative Λ, as these have applications in mathematical physics, such as in string theory.

The anti-de Sitter metric may be written in spherical polar coordinates as

$$ds^2 = -f dt^2 + f^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \quad \text{where} \quad f = \left(1 - \frac{\Lambda r^2}{3}\right) \quad \text{and} \quad \Lambda < 0 \quad (167)$$

This metric actually also gives the de Sitter spacetime if the cosmological constant is taken to be positive, but these coordinates does not cover the entire spacetime in that case. In ordinary fashion, we let $\mathcal{S}$ be the spheres of constant radius $r = R$ and constant time $t = \tau$, and find the normal vectors

$$t_\mu = -\sqrt{f}\delta^t_\mu \quad \text{and} \quad n_\mu = \frac{1}{\sqrt{f}}\delta^r_\mu \quad \Longrightarrow \quad k_{\pm\mu} = t_\mu \pm n_\mu = -\sqrt{f}\delta^t_\mu + \frac{1}{\sqrt{f}}\delta^r_\mu \quad (168)$$

This allows us to calculate the second fundamental forms of the surface:

$$K_{ij}(k_\pm) = -k_{\mu\pm}\left(\frac{\partial^2 x^\mu}{\partial u^i \partial u^j} + \Gamma^\mu_{\alpha\beta}\frac{\partial x^\alpha}{\partial u^i}\frac{\partial x^\beta}{\partial u^j}\right) \quad (169)$$

The derivatives are trivial since the surface is defined by constant coordinates r and t. One only needs to insert the Christoffel symbols, which gives

$$K_{\theta\theta}(k_\pm) = -k_{\mu\pm}\Gamma^\mu_{\theta\theta} = \pm R\sqrt{1 - \frac{\Lambda R^2}{3}} \quad \text{and} \quad K_{\varphi\varphi}(k_\pm) = -k_{\mu\pm}\Gamma^\mu_{\varphi\varphi} = K_{\theta\theta}(k_\pm)\sin^2\theta \quad (170)$$

The null expansions are thus

$$\theta_\pm = K_{ij}(k_\pm)\gamma^{ij} = K_{\theta\theta}(k_\pm)\left(\frac{1}{R^2} + \frac{\sin^2\theta}{R^2\sin^2\theta}\right) = \pm\frac{2}{R}\sqrt{1 - \frac{\Lambda R^2}{3}} \quad (171)$$

So that the Hawking mass of the sphere becomes

$$\begin{aligned} M_H &= \sqrt{\frac{A}{16\pi}}\left(1 - \frac{1}{16\pi}\oint \frac{4}{R^2}\left(1 - \frac{\Lambda R^2}{3}\right)dS\right) = \sqrt{\frac{A}{16\pi}}\left(\underbrace{1 - \frac{1}{4\pi}\oint \frac{1}{R^2}dS}_{=0} + \frac{1}{4\pi}\oint \frac{\Lambda}{3}dS\right) \\ &= \sqrt{\frac{A}{16\pi}}\frac{1}{16\pi}\oint \frac{4\Lambda}{3}dS \end{aligned} \quad (172)$$

Which is directly proportional to the cosmological constant. Thus, the Hawking mass is strictly negative in anti-de Sitter, and positive in regular de Sitter (for surfaces in the covered patch). In anti-de Sitter, the Hawking mass of large asymptotic spheres evidently diverges, since the it is also proportional to the area of the surface, which has no upper bound. On the other hand, the regular de Sitter spacetime is closed, which means that surfaces cannot be arbitrarily large, so the Hawking mass is bounded in that case.

It seems more sensible to have a definition of quasi-local mass that is finite if the amount of 'regular' mass in the spacetime is finite, even if the cosmological constant is non-zero. This is only possible (with a negative cosmological constant) if the contribution from the dark energy is excluded. This motivates the correction

$$M_H = \sqrt{\frac{A}{16\pi}}\left(1 + \frac{1}{16\pi}\oint \left(\theta_+\theta_- - \frac{4\Lambda}{3}\right)dS\right) \quad (173)$$

the result of which is that the Hawking mass is identically zero for spheres in both Minkowski and anti-de Sitter (de Sitter). In a sense, this choice has to do with whether or not the dark energy that the cosmological constant represents should be interpreted as a mass or something else entirely.

6.3 Modified Hawking mass of spheres in AdS–Schwarzschild

Now that the Hawking mass has been modified to yield zero mass for spheres in a vacuum spacetime with cosmological constant, it would also be desirable if it produced the correct mass for spheres in a spacetime with both mass and a non-zero cosmological constant. A simple example of such a spacetime is the AdS-Schwarzschild spacetime, which, as its name suggests, is the AdS spacetime with a black hole in it. The metric of this spacetime is

$$ds^2 = -f dt^2 + f^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \quad \text{where} \quad f = \left(1 - \frac{\Lambda r^2}{3} - \frac{2M}{r}\right) \tag{174}$$

The calculation in this spacetime will be nearly identical to the one in regular AdS, so not much additional work is required. The only difference which needs to be taken into account is the Christoffel symbols, but in terms of f these are actually the same. Therefore, we shall skip ahead and state directly the null expansions:

$$\theta_\pm = \pm\frac{2}{R}\sqrt{1 + \frac{R^2}{a^2} - \frac{2M}{R}} \tag{175}$$

This gives us the Hawking mass of a sphere as:

$$\begin{aligned} M_H &= \sqrt{\frac{A}{16\pi}}\left(1 - \frac{1}{16\pi}\oint\left(\frac{4}{R^2}\left(1 + \frac{R^2}{a^2} - \frac{2M}{R}\right) - \frac{\Lambda}{3}\right)dS\right) \\ &= \sqrt{\frac{A}{16\pi}}\left(\frac{1}{4\pi}\oint\left(\underbrace{-\frac{1}{a^2} - \frac{\Lambda}{3}}_{=0} + \frac{2M}{R^3}\right)dS\right) = \sqrt{\frac{A}{16\pi}}\frac{1}{4\pi R^2}\frac{2M}{R}4\pi R^2 = M \end{aligned} \tag{176}$$

This means that the contribution from the cosmological constant is the same in both cases. Considering the fact that the field equations are non-linear, we find it somewhat surprising that this happens; but in any case, it is a fortunate result, since now the Hawking mass works precisely the same in spacetimes with non-zero cosmological constant as in those without one.

6.4 Monotonicity of the Geroch mass

Since the Geroch mass coincides with the Hawking mass if the foliation is chosen suitably, applying the same modification to the Geroch mass should make it well-behaved in spacetimes with non-zero cosmological constant as well; at the very least in the cases when they coincide. It is worth verifying that such a modification preserves the monotonicity property of the Geroch mass. Modified for Λ, the Geroch mass would read

$$M_G = \sqrt{\frac{A}{16\pi}}\left(1 - \frac{1}{16\pi}\oint\left(p^2 + \frac{4\Lambda}{3}\right)dS\right) \tag{177}$$

Showing that this is monotone requires modifying the proof in section 5.1 to include the last term. Since the steps will be nearly identical, we shall only state the highlights. We begin by writing

$$M_G = \sqrt{\frac{A}{16\pi}}\frac{1}{16\pi}\underbrace{\oint\left(2R_S - p^2 - \frac{4\Lambda}{3}\right)d\mathcal{S}}_{=W+Q} \tag{178}$$

Where W is the same as in the original proof. Then, we find that

$$\dot{Q} = -\oint \frac{4\Lambda}{3} d\dot{\mathcal{S}} = -\oint \frac{4\Lambda}{3} d\mathcal{S} = Q \quad \text{and} \quad \dot{M}_G = \sqrt{\frac{A}{16\pi}} \left(\frac{W}{2} + \dot{W} + \frac{Q}{2} + \dot{Q} \right) \tag{179}$$

Recall the derivative of W:

$$\dot{W} = \oint \left(R_\Sigma + U^2 \right) dS - \frac{1}{2} W \tag{180}$$

In the case of a non-zero cosmological constant, Gauss' Theorema Egregium includes a contribution from Λ,

$$R_\Sigma = 2\mu + 2\Lambda + \kappa_{ab}\kappa^{ab} - \kappa^2 \tag{181}$$

so that, for a maximal hypersurface where $\kappa = 0$,

$$\dot{W} = \oint \left(2\mu + \kappa_{ab}\kappa^{ab} + U^2 \right) dS - \frac{1}{2} W + \oint 2\Lambda d\mathcal{S} \tag{182}$$

The result is

$$\dot{M}_G = \sqrt{\frac{A}{16\pi}} \left(\underbrace{\oint \left(2\mu + \kappa_{ab}\kappa^{ab} + U^2 \right) dS}_{\geq 0} + \underbrace{\frac{3}{2} Q + \oint 2\Lambda d\mathcal{S}}_{=0} \right) \tag{183}$$

We see that it was precisely this modification that was necessary in order to preserve the monotonicity property of the Geroch mass for non-zero Λ[30]. This further shows that subtracting the cosmological constant is a reasonable thing to do for these mass definitions.

6.5 Hawking mass in FLRW spacetimes

The Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime is the standard model of modern cosmology, describing an isotropic and homogeneous expanding (or contracting) universe. It has a cosmological constant that may be either positive or negative, and as previously stated, current observations suggest that it should be positive. This makes it a good candidate to study the Hawking mass in.

The FLRW metric is given by

$$ds^2 = -dt^2 + a^2(t) \left(\frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2\theta d\varphi^2) \right) \tag{184}$$

where $a(t)$ is the scale factor, which governs the expansion of the universe, and k is a parameter that determines the constant curvature, which may be anything between -1 and 1. A negative value of k corresponds to an open universe, in which two initially parallel geodesics will diverge from each other. Likewise, a positive value gives a closed universe, such that initially parallel geodesics tend to converge. The case $k = 0$ corresponds to a completely flat universe, which is what our universe is believed to be.

We will yet again regard the mass of a sphere of constant $r = R$. The calculation is very similar to earlier sections, the only differences being the normals, which are now

$$t^\mu = \delta^\mu_t \quad \text{and} \quad n^\mu = \delta^\mu_r \frac{a(t)}{\sqrt{1 - kr^2}} \quad \text{and} \quad k^\mu_\pm = t^\mu \pm n^\mu \tag{185}$$

[30] If Λ is positive, then the Geroch mass will be monotone regardless, but removing the contribution from Λ makes the definition good in all cases.

and the Christoffel symbols, which may be found in any standard reference, such as [15]. Putting everything together, we find that

$$K_{\theta\theta}(k_\pm) = -k_{\mu\pm}\Gamma^{\mu}_{\theta\theta} = ar(\dot{a}r \pm \sqrt{1-kr^2}) \quad \text{and} \quad K_{\varphi\varphi}(k_\pm) = -k_{\mu\pm}\Gamma^{\mu}_{\varphi\varphi} = K_{\theta\theta}(k_\pm)\sin^2\theta \tag{186}$$

Thus

$$\theta_\pm = \frac{2a}{r}(\dot{a}r \pm \sqrt{1-kr^2}) \quad \Longrightarrow \quad \theta_+\theta_- = 4a^2\left(\dot{a}^2 + k - \frac{1}{r^2}\right) \tag{187}$$

The Hawking mass is then

$$\begin{aligned} M_H &= \sqrt{\frac{A}{16\pi}}\left(1 + \frac{1}{16\pi}\oint 4a^2\left(\dot{a}^2 + k - \frac{1}{R^2}\right)dS\right) \\ &= \sqrt{\frac{A}{16\pi}}\left(\frac{1}{4\pi}\oint a^2\left(\dot{a}^2 + k\right)dS\right) = \frac{R^3}{2}a^2(\dot{a}^2 + k) \end{aligned} \tag{188}$$

Which has the proper dimensionality since $\dot{a}^2$ and k have units of inverse length squared. This is evidently positive if $\dot{a}^2 + k \geq 0$, but could in principle be negative. This statement can be made more precise if one invokes the Friedmann equation[31]

$$\dot{a}^2 = a^2\left(\frac{8\pi\rho + \Lambda}{3}\right) - k \tag{189}$$

Where ρ is the mass density of matter in the universe. From this, one can immediately tell that

$$8\pi\rho + \Lambda \geq 0 \quad \Longrightarrow \quad \dot{a}^2 + k \geq 0 \quad \Longrightarrow \quad M_H \geq 0 \tag{190}$$

This result makes sense: as long as the regular matter dominates over Λ, the Hawking mass of spheres will not be negative.

[31] Which is derived by inserting the FLRW metric into Einstein's field equations.

7 Light cone cuts

It is clear that the Hawking mass is a suitable definition of mass in some special cases, such as when considering the mass contained in spheres in a spherically symmetric spacetime, but not for the more general case of an abitrary surface, even in spherically symmetric spacetimes. The main problem is that the Hawking mass becomes negative for most non-spherical surfaces, which is bad news for a mass; but if one knew the exact circumstances under which this happens (or rather, doesn't happen), one might be able to make a modification to the Hawking mass so that it is positive for all surfaces. Alternatively, one could place a restriction on the type of surfaces that are allowed.

As it turns out, there is a quite large class of surfaces for which the Hawking mass works well, at the very least in vacuum spacetimes. We have managed to identify a large class of surfaces for which the Hawking mass is positive, and which has a very simple description. These surfaces are the intersections of the light cone of an intertial observer with arbitrary spatial hypersurfaces, also called light cone cuts.

What makes such a surface special is that, in principle, *the entire surface can be seen simultaneously from a single point*. Parts of the surface that are farther away in space are closer in time, so that if it were to emit light, all of it would arrive at the observer simultaneously. Hence, these surfaces are what we see when we look around us.

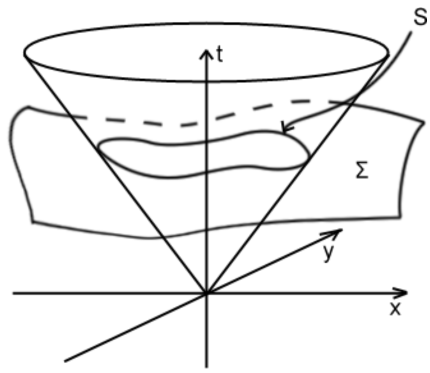

Figure 9: The surface Σ intersects the light cone, yielding a curve $\mathcal{S}$ which is the analogue of a light cone cut in 2+1 dimensions.

7.1 Light cone cuts in Minkowski

We shall now demonstrate that the surface of intersection between the light cone and an arbitrary spatial hypersurface has a Hawking mass that is identically zero in the Minkowski spacetime. The line element is

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{191}$$

The light cone with respect to the origin consists of all points whose separation from the origin is null. In these coordinates, this means that $r = t$. Setting $dr = dt$ yields the line element on the light cone:

$$ds^2 = r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{192}$$

Keep in mind that the light cone is a three-dimensional null hypersurface, and not a two-surface. Specifying a spacelike hypersurface amounts to assigning a time to each point in space: $t = t(x)$. The cut surface $\mathcal{S}$ is then given by restricting $r = t(x)$, so that, in terms of the remaining free coordinates, one can write

$$\begin{aligned} t &= \Omega(\theta, \varphi) \\ r &= \Omega(\theta, \varphi) \end{aligned} \tag{193}$$

the result of which is that we have selected a radius and time for each direction on the sky. In regular fashion, we shall use $u^i = (u, v)$ as coordinates on $\mathcal{S}$, and the simplest possible choice is to set $u = \theta$ and $v = \varphi$. Thus, the inner metric on the cut surface can be written

$$ds^2 = \Omega^2(du^2 + \sin^2 u dv^2) \tag{194}$$

which makes it evident that any such surface is conformal to a sphere. To calculate the Hawking mass of this surface, we must find the null expansions. This will be slightly more complicated in this case, since we do not know the exact specification of the surface. Therefore, we will use a more general procedure to solve for the null normals of the surface.

It is easy to find one null normal of $\mathcal{S}$, namely the normal of the light cone[32]. Since $\mathcal{S}$ is contained entirely within the light cone, it must share this normal vector:

$$k_+^\mu = \delta_t^\mu + \delta_r^\mu = \begin{pmatrix} 1, & 1, & 0, & 0 \end{pmatrix} \tag{195}$$

Finding a second null normal will not be quite as easy, but we can do it by first solving for the tangent vectors of the surface. We have

$$e_u^\mu = \frac{\partial x^\mu}{\partial u} = \begin{pmatrix} \Omega_u, & \Omega_u, & 1, & 0 \end{pmatrix} \equiv m^\mu \tag{196}$$

$$e_v^\mu = \frac{\partial x^\mu}{\partial v} = \begin{pmatrix} \Omega_v, & \Omega_v, & 0, & 1 \end{pmatrix} \equiv n^\mu \tag{197}$$

where we have used the short-hand notation Ω_u and Ω_v to denote the partial derivatives of Ω with respect to u and v. Now, we may solve for the second null normal by setting up a system of equations: the second normal must be orthogonal to the tangent vectors, and its product with the other normal must be -2.

$$\begin{matrix} m^\mu k_{-\mu} = 0 \\ n^\mu k_{-\mu} = 0 \\ k_+^\mu k_{-\mu} = -2 \end{matrix} \qquad \Longrightarrow \qquad \begin{matrix} \Omega_u(k_{-0} + k_{-1}) + k_{-2} = 0 \\ \Omega_v(k_{-0} + k_{-1}) + k_{-3} = 0 \\ k_{-0} + k_{-1} = -2 \end{matrix} \tag{198}$$

Which has the solution

$$k_{-\mu} = \begin{pmatrix} k_{-0}, & -2 - k_{-0}, & 2\Omega_u, & 2\Omega_v \end{pmatrix} \tag{199}$$

Finally, we impose that this vector should be null:

$$k_{-\mu} k_{-\nu} g^{\mu\nu} = 0 \qquad \Longrightarrow \qquad -k_{-0}^2 + (2 + k_{-0})^2 + \frac{1}{r^2} 4\Omega_u^2 + \frac{1}{r^2 \sin^2\theta} 4\Omega_v^2 = 0 \tag{200}$$

which gives

$$k_{-0} = -\left(1 + \frac{1}{r^2}\Omega_u^2 + \frac{1}{r^2 \sin^2\theta}\Omega_v^2\right) = -\left(1 + \frac{1}{\Omega^2}\Omega_u^2 + \frac{1}{\Omega^2 \sin^2 u}\Omega_v^2\right) \tag{201}$$

on the surface, which is where this vector is defined. A trained eye recognizes that the above expression can be written in terms of the covariant derivative on the unit sphere:

$$k_{-0} = -\left(1 + \frac{1}{\Omega^2}(\nabla\Omega)^2\right) = -\left(1 + \frac{1}{\Omega^2}\gamma^{ij}\nabla_i\Omega\nabla_j\Omega\right) \tag{202}$$

where γ is the metric on the unit sphere, and ∇ is the covariant derivative with respect to this metric. For clarity,

$$\gamma_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 u \end{pmatrix} \tag{203}$$

Defining $\alpha = \frac{(\nabla\Omega)^2}{\Omega^2}$, we have

$$k_{-\mu} = \begin{pmatrix} -1 - \alpha, & -1 + \alpha, & 2\Omega_u, & 2\Omega_v \end{pmatrix} \tag{204}$$

[32] Which is also tangential to the light cone, which is a somewhat counterintuitive feature of null surfaces.

Now we may calculate the second fundamental forms and the null expansions. One the null expansions is very simple to calculate (it goes much like the earlier calculations of null expansions), due to the trivial form of $\vec{k}_+$. We shall thereby only state the result:

$$\theta_+ = \frac{2}{\Omega} \tag{205}$$

The other null expansion becomes more involved. We begin by making the observation that the metric on the surface is diagonal, which means that only the diagonal elements of $K_{ij}(\vec{k}_-)$ are required to find the null expansion. Setting $i = j$ in the formula (22) for the second fundamental form, and inserting $\vec{k}_-$, gives

$$K_{ii}(\vec{k}_-) = 2\frac{\partial^2\Omega}{\partial u^{i2}} - \Big((\alpha-1)\Gamma^r_{\alpha\beta} + 2\Omega_u\Gamma^\theta_{\alpha\beta} + 2\Omega_v\Gamma^\varphi_{\alpha\beta}\Big)\frac{\partial x^\alpha}{\partial u^i}\frac{\partial x^\beta}{\partial u^i} \tag{206}$$

Removing all Christoffel symbols that are zero, and taking into account the derivatives that vanish, the remainder is

$$\begin{aligned} K_{ii}(\vec{k}_-) = 2\frac{\partial^2\Omega}{\partial u^{i2}} - \Bigg[(\alpha-1)&\left(\Gamma^r_{\theta\theta}\left(\frac{\partial\theta}{\partial u^i}\right)^2 + \Gamma^r_{\varphi\varphi}\left(\frac{\partial\varphi}{\partial u^i}\right)^2\right) \\ +2\Omega_u&\left(2\Gamma^\theta_{r\theta}\frac{\partial r}{\partial u^i}\frac{\partial\theta}{\partial u^i} + \Gamma^\theta_{\varphi\varphi}\left(\frac{\partial\varphi}{\partial u^i}\right)^2\right) \\ &+2\Omega_v\left(2\Gamma^\varphi_{r\varphi}\frac{\partial r}{\partial u^i}\frac{\partial\varphi}{\partial u^i}\right)\Bigg] \end{aligned} \tag{207}$$

Now one only needs to insert the Christoffel symbols and the derivatives:

$$K_{uu}(\vec{k}_-) = 2\frac{\partial^2\Omega}{\partial u^2} + \left[\Omega(\alpha-1) - 4\Omega_u^2\frac{1}{\Omega}\right] \tag{208}$$

$$K_{vv}(\vec{k}_-) = 2\frac{\partial^2\Omega}{\partial v^2} + \left[\Omega\sin^2 u(\alpha-1) + 2\Omega_u\sin u\cos u - 4\Omega_v^2\frac{1}{\Omega}\right] \tag{209}$$

The remaining null expansion can then be assembled:

$$\begin{aligned} \theta_- &= \frac{1}{\Omega^2}\left(\underbrace{2\frac{\partial^2\Omega}{\partial u^2} + \frac{2}{\sin^2 u}\frac{\partial^2\Omega}{\partial v^2} + 2\Omega_u\cot u}_{=2\nabla^2\Omega} + \left[\Omega(\alpha-1) - 4\Omega_u^2\frac{1}{\Omega}\right] + \left[\Omega(\alpha-1) - 4\Omega_v^2\frac{1}{\Omega\sin^2 u}\right]\right) \\ &= \frac{1}{\Omega^2}\left(2\nabla^2\Omega + 2\Omega(\alpha-1) - \frac{4}{\Omega}\underbrace{\left[\Omega_u^2 + \frac{1}{\sin^2\theta}\Omega_v^2\right]}_{(\nabla\Omega)^2}\right) = \frac{2}{\Omega^2}\left(\Omega(\alpha-1) + \nabla^2\Omega - \frac{2(\nabla\Omega)^2}{\Omega}\right) \end{aligned} \tag{210}$$

Where we, again, recognized the covariant derivative on the unit sphere, both as the squared gradient and the Laplacian. Since $(\nabla\Omega)^2/\Omega = \Omega\alpha$, we can write

$$\theta_- = -\frac{2}{\Omega}\Big(1 + \alpha - \frac{\nabla^2\Omega}{\Omega}\Big) \tag{211}$$

We may finally calculate the Hawking mass of the surface:

$$M_H = \sqrt{\frac{A}{16\pi}}\left(1 + \frac{1}{16\pi}\oint\theta_+\theta_-\Omega^2 dS_2\right) \tag{212}$$

Since the surface element is $\Omega^2 dS_2$, where dS_2 is the area element of the unit 2-sphere. Thus

$$M_H = \sqrt{\frac{A}{16\pi}} \left(1 - \frac{1}{4\pi} \oint \left(1 + \alpha - \frac{\nabla^2 \Omega}{\Omega}\right) dS_2\right) = \sqrt{\frac{A}{16\pi}} \left(\frac{1}{4\pi} \oint \left(\alpha - \frac{\nabla^2 \Omega}{\Omega}\right) dS_2\right) \tag{213}$$

Now note that

$$\frac{\nabla^2 \Omega}{\Omega} = \frac{\nabla(\nabla\Omega)}{\Omega} = \nabla\left(\frac{\nabla\Omega}{\Omega}\right) - \nabla\Omega\nabla\frac{1}{\Omega} = \nabla\left(\frac{\nabla\Omega}{\Omega}\right) + \underbrace{\nabla\Omega\frac{1}{\Omega^2}\nabla\Omega}_{\alpha} \tag{214}$$

The result is

$$M_H = -\sqrt{\frac{A}{16\pi}} \frac{1}{4\pi} \oint \nabla\left(\frac{\nabla\Omega}{\Omega}\right) dS_2 = 0 \tag{215}$$

Since we are integrating a total derivative over a closed surface. We can conclude that any surface defined in this way has identically zero Hawking mass in the Minkowski spacetime. If one could quantify the difference between these surfaces and the more general surfaces for which the Hawking mass is negative, it could be possible to identify a modification of the Hawking mass that makes it work well for any surface in Minkowski. Alternatively, it might just be sensible to restrict the surfaces one allows when regarding the Hawking mass (but for what reason is not clear).

7.2 Light cone cuts in Einstein's static universe

The natural next step is to investigate how general the above result is. Does it hold in any spacetime, even those with matter content? For spacetimes with non-zero or non-constant curvature, it becomes significantly harder to characterize the light cone and its intersections with hypersurfaces. The light cone tends to get very contrived in the presence of curvature. Take for example the Schwarzschild spacetime. It is a well known fact that black holes curve spacetime to such an extent that they act as a 'gravitational lens' for light. Light can, for example, be bent around the black hole in such a way that multiple images of the same object may be seen. It is thus clear that light cones in such a spacetime can exhibit some very complicated behaviour, such as self-intersections.

In the case of a spacetime with a non-zero constant curvature, things are not quite as complicated, however. Light cones in some of these spacetimes are quite well-behaved and easy to characterize. Depending on one's point of view, such a spacetime may also be considered to contain a constant mass density (the cosmological constant, as discussed earlier), which can contribute to the Hawking mass, unless mitigated by modifications.

With this in mind, we shall investigate how the Hawking mass works out for light cone cuts in a slightly less trivial spacetime than Minkowski, namely the static Einstein universe. This is the solution that Einstein originally produced by inserting the cosmological constant, with the goal of producing a spacetime that does not expand or contract. This spacetime is a special case of the FLRW spacetime that we have discussed earlier, with a constant matter energy density acting as the cosmological constant.

The line element in Einstein's static universe is

$$ds^2 = -dt^2 + dr^2 + \sin^2 r(d\theta^2 + \sin^2\theta d\varphi^2) \tag{216}$$

As the form of the metric suggests, the radial coordinate r is a periodic coordinate. The Einstein universe can be understood as the surface of a three-dimensional sphere, plus time, so that motion in any direction will eventually lead back to the same point. In this sense, this is a closed universe. The light cone is given by $r = t$ in this spacetime as well, since it is spherically symmetric. For this reason, most of the calculations from Minkowski will carry over directly, so we will not state them all in full. The line element on the light cone is here

$$ds^2 = \sin^2 r(d\theta^2 + \sin^2\theta d\varphi^2) \tag{217}$$

Notice especially the fact that the light cone metric is singular at $r = n\pi$. This singularity is due to the choice of coordinates, just like the singular behaviour of the two-sphere metric at the poles. An arbitrary light cone cut $\mathcal{S}$ may again be specified by selecting $r = t = \Omega(\theta, \varphi)$. By the same procedure as above, we find null normals of this surface on the same form as in Minkowski:

$$k_{+\mu} = \begin{pmatrix} 1, & 1, & 0, & 0 \end{pmatrix}, \quad k_{-\mu} = \begin{pmatrix} -1-\alpha, & \alpha - 1, & 2\Omega_u, & 2\Omega_v \end{pmatrix} \tag{218}$$

The only difference being that $\alpha = (\nabla\Omega)^2/\sin^2\Omega$ this time (with ∇ still being the covariant derivative on the unit sphere). The outward null expansion is, again, simple to calculate.

$$\theta_+ = \frac{2\sin\Omega\cos\Omega}{\sin^2\Omega} = 2\cot\Omega \tag{219}$$

We see that it has a similar form to the same null expansion in Minkowski. For the other null expansion, we follow the same procedure as in Minkowski and find that

$$K_{uu}(\vec{k}_-) = 2\frac{\partial^2\Omega}{\partial u^2} + \left[(\alpha - 1)\sin\Omega\cos\Omega - 4\Omega_u^2\cot\Omega\right] \tag{220}$$

$$K_{vv}(\vec{k}_-) = 2\frac{\partial^2\Omega}{\partial v^2} + \left[(\alpha - 1)\sin\Omega\cos\Omega\sin^2 u + 2\Omega_u\sin u\cos u - 4\Omega_v^2\cot\Omega\right] \tag{221}$$

So that, after some simplification

$$\theta_- = \frac{2}{\sin^2\Omega}\left[\nabla^2\Omega - (1+\alpha)\sin\Omega\cos\Omega\right] \tag{222}$$

We can now calculate the Hawking mass:

$$M_H = \sqrt{\frac{A}{16\pi}}\left(1 + \frac{1}{4\pi}\oint\cot\Omega\left[\nabla^2\Omega - (1+\alpha)\sin\Omega\cos\Omega\right]dS_2\right) \tag{223}$$

Where we used the fact that the surface element is $\sin^2\Omega dS_2$, where dS_2 is the surface element on the unit two-sphere. Now, simplifying this expression requires some derivative gymnastics. Observe:

$$\nabla(\ln(\sin^2\Omega)) = \frac{1}{\sin^2\Omega}\nabla\sin^2\Omega = \frac{1}{\sin^2\Omega}2\sin\Omega\cos\Omega\nabla\Omega = 2\cot\Omega\nabla\Omega \tag{224}$$

From this we can conclude that

$$\frac{1}{2}\nabla^2(\ln(\sin^2\Omega)) = \nabla(\cot\Omega\nabla\Omega) = -\frac{(\nabla\Omega)^2}{\sin^2\Omega} + \cot\Omega\nabla^2\Omega \tag{225}$$

Which gives us

$$-\cot\Omega\left(\nabla^2\Omega - \frac{(\nabla\Omega)^2}{\sin^2\Omega\cot\Omega}\right) = \frac{1}{2}\nabla^2(\ln(\sin^2\Omega)) \tag{226}$$

Our integrand contains the terms $\cot\Omega\Big[\nabla^2\Omega - \alpha\sin\Omega\cos\Omega\Big]$, which we can rewrite using the above result:

$$\cot\Omega\Big[\nabla^2\Omega - \alpha\sin\Omega\cos\Omega\Big] = \cot\Omega\Big[\nabla^2\Omega - \frac{(\nabla\Omega)^2}{\sin^2\Omega\cot\Omega}\cos^2\Omega\Big] \tag{227}$$

$$= \cot\Omega\Big[\nabla^2\Omega - \frac{(\nabla\Omega)^2}{\sin^2\Omega\cot\Omega} + \frac{(\nabla\Omega)^2}{\cot\Omega}\Big] = -\frac{1}{2}\nabla^2(\ln(\sin^2\Omega)) + (\nabla\Omega)^2 \tag{228}$$

The expression for the Hawking mass thereby becomes

$$\begin{aligned} M_H &= \sqrt{\frac{A}{16\pi}}\left(1 + \frac{1}{4\pi}\oint\Bigg[-\cot\Omega\sin\Omega\cos\Omega + (\nabla\Omega)^2\Bigg]dS_2\right) \\ &= \sqrt{\frac{A}{16\pi}}\left(\frac{1}{4\pi}\oint\Bigg[\sin^2\Omega + (\nabla\Omega)^2\Bigg]dS_2\right) \geq 0 \end{aligned} \tag{229}$$

But recall that $\sin^2\Omega dS_2 = dS$. Thus,

$$M_H = \sqrt{\frac{A}{16\pi}}\frac{1}{4\pi}\left(\oint dS + \oint(\nabla\Omega)^2 dS_2\right) = \sqrt{\frac{A}{16\pi}}\frac{1}{4\pi}\left(A + \oint(\nabla\Omega)^2 dS_2\right) \tag{230}$$

Which is positive for all surfaces. Notice how this expression seems to have the wrong dimensionality. This stems from the fact that the radial coordinate is angular, which means that it is dimensionless. The metric really contains a factor R_C (the radius of curvature), with units of length, which has been set to one:

$$ds^2 = -dt^2 + R_C^2\big(dr^2 + \sin^2 r(d\theta^2 + \sin^2\theta d\varphi^2)\big) \tag{231}$$

Dimensional analysis implies that

$$M_H = \sqrt{\frac{A}{16\pi}}\frac{1}{4\pi R_C^2}\left(A + \oint(\nabla\Omega)^2 dS_2\right) \tag{232}$$

if $R_C \neq 1$. This length factor sets the curvature scale of the universe, and thereby the cosmological constant; in fact, $\Lambda = 1/R_C^2$. A very small R_C means that the constant curvature is large, which means a large cosmological constant, or, equivalently, a large matter density. This is reflected by the Hawking mass: the amount of mass contained increases as the curvature increases.

While this mass is positive, it does not appear to be monotone in a strict sense. For example: given a sphere, the integral $\oint(\nabla\Omega)^2 dS_2$ is identically zero. It is always possible to construct a surface contained entirely inside a sphere which has a larger area than the sphere. Such a surface cannot be a perfect sphere, so its gradient must be atleast partially non-zero. Thus, both parts that go into the Hawking mass are larger for the contained surface than the exterior sphere, so that the smaller of the two surfaces has the larger mass.

8 Discussion

We have herein calculated a few explicit examples of the Hawking mass, which demonstrates the behaviour of this definition, mainly in the Minkowski and Schwarzschild spacetimes. It is shown how non-spherical surfaces tend to make the Hawking mass negative in the Minkowski spacetime, which demonstrates that the Hawking mass is affected by the shape of the surface and not just its contents; an ideal definition would be identically zero for all surfaces in Minkowski. In the Schwarzschild spacetime it is not calculated for anything more general than a sphere, but it is reasonable to believe that the choice of a non-spherical surface would result in a smaller than expected mass in this spacetime, considering the above.

We then identified a large class of surfaces, the light cone cuts, for which the Hawking mass is identically zero in Minkowski, and strictly positive for a spacetime with positive cosmological constant. Thus, in Minkowski, the Hawking mass is independent of the choice of surface within the class. In other words, these surfaces lack the specific property which tends to make the Hawking mass negative. Further study of these surfaces might make it possible to quantify this property, which would allow one to state a modified version of the Hawking mass where the contribution from this property has been removed, which would presumably yield more consistent results than the Hawking mass.

It would be interesting to work out how the Hawking mass of light cone cuts behaves in spacetimes that are not spherically symmetric. An attempt was made to examine this in the Kasner spacetime, which is a spacetime that expands and contracts in a non-isotropic way. This calculation turned out to be extremely heavy and time-consuming, and a proper analysis will probably require more sophisticated methods than used herein. It would also be interesting to look at light cone cuts in the Schwarzschild spacetime, but since the light cone lacks a simple description there, this is also expected to require more sophisticated methods.

The Geroch mass, on the other hand, was not originally intended to be a satisfactory definition of quasi-local mass. This is made evident in the calculations we have made, which illustrates how this definition depends a lot upon the choice of spacetime foliation. However, as is also demonstrated, the Geroch mass has its uses: it can be made monotone under the inverse mean curvature flow by a specific choice of foliation. This made it useful in proving the positive energy theorem and the Riemannian Penrose inequality, and it may be useful in proving other geometric theorems (as may the Hawking mass). The inverse mean curvature flow was also studied in great detail, both in theory and numerically, explicitly demonstrating some of the hardships that must be overcome to solve geometric flow equations of this kind.

In addition, the contribution of a non-zero cosmological constant was reflected over, and modifications of the Hawking and Geroch masses were provided, which makes them equally functional in spacetimes where $\Lambda \neq 0$ as in spacetimes where $\Lambda = 0$. Whether or not such a modification should be made to a quasi-local definition of mass depends on whether one wishes to treat the 'dark energy' as a real mass or not, but it seems sensible to do so since it makes the Hawking and Geroch masses well-defined in the limit of large asymptotic spheres.

The numerical evaluation of inverse mean curvature flow was an excellent way to gain more insight into geometric flows and the challenges in dealing with these. The algorithm seems to work well for its purpose, but there is a lot more that could be done in the analysis of it, such as a more in depth stability analysis and convergence analysis. It was decided that these things were somewhat outside of the scope for this report, so they were not pursued further. A completely rigourous treatment of all aspects of such an algorithm could consitute a full thesis on its own.

A further study of quasi-local mass might also look at other definitons, such as those suggested by Wand and Yau [12] or Penrose [11]. The definition suggested by Yau and Wang supposedly has all the properties required of a quasi-local mass, but this definition is mathematically very sophisticated and not very explicit, which makes its connection to physical notions rather obscure. It would be of interest to see how this definition of mass can be interpreted from a physical point of view.

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9 Appendix A - Christoffel symbols of Schwarzschild in trumpet coordinates

In the trumpet coordinates (t, r, θ, φ), where r is measured from an inner sphere of area $4\pi R_0$ so that a radius r corresponds to a sphere of area $4\pi(R_0 + r)$, the Schwarzschild spacetime is described by the line element

$$ds^2 = -f dt^2 + \frac{2f_1}{r} dt dr + f_2^2 (dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2) \tag{233}$$

where

$$f = \left(1 - \frac{2M}{r + R_0}\right), \quad f_1 = \sqrt{2r(M - R_0) + R_0(2M - R_0)}, \quad f_2 = 1 + \frac{R_0}{r}, \quad (0 < R_0 \leq M) \tag{234}$$

The Christoffel symbols for these coordinates are not easily found in any standard reference, so we will give them here

$$\Gamma^t_{\alpha\beta} = \begin{pmatrix} \frac{Mf_1}{r(r+R_0)^2} & \frac{M}{r^2} & 0 & 0 \\ \frac{M}{r^2} & \frac{f_2(f_1^2 - R_0^2)}{r^2 f_1} & 0 & 0 \\ 0 & 0 & -f_1 f_2 & 0 \\ 0 & 0 & 0 & -f_1 f_2 \sin^2\theta \end{pmatrix} \tag{235}$$

$$\Gamma^r_{\alpha\beta} = \begin{pmatrix} \frac{Mf}{(r+R_0)^2} & -\frac{Mf_1}{r(r+R_0)^2} & 0 & 0 \\ -\frac{Mf_1}{r(r+R_0)^2} & -\frac{M}{r^2} & 0 & 0 \\ 0 & 0 & -(r+R_0)f & 0 \\ 0 & 0 & 0 & -(r+R_0)f\sin^2\theta \end{pmatrix} \tag{236}$$

$$\Gamma^\theta_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r+R_0} & 0 \\ 0 & \frac{1}{r+R_0} & 0 & 0 \\ 0 & 0 & 0 & -\cos\theta\sin\theta \end{pmatrix}, \quad \Gamma^\varphi_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r+R_0} \\ 0 & 0 & 0 & \cot\theta \\ 0 & \frac{1}{r+R_0} & \cot\theta & 0 \end{pmatrix} \tag{237}$$

10 Appendix B - Useful geometric theorems

10.1 Gauss' theorema egregium

To understand how the geometry of hypersurfaces is related to their mass content, we shall require Gauss' theorema egregium (remarkable theorem). This theorem connects the intrinsic and extrinsic curvatures of a surface, and is therefore very useful in geometry. Gauss himself considered regular two-dimensional surfaces in a three-dimensional manifold, but we shall state the theorem a bit more generally.

Consider a $(n-1)$ dimensional hypersurface $\mathcal{S}$, embedded in a manifold Σ of dimension n equipped with a metric g_{ab} and its associated covariant derivative ∇_a such that $\nabla_a g_{bc} = 0$. $\mathcal{S}$ then has the induced metric $\gamma_{AB} = e^a{}_A e^b{}_B g_{ab}$ and an associated covariant derivative D_A such that $D_A \gamma_{BC} = 0$. Notice the fact that we have now chosen to use capital indices on tensors in the tangent space of $\mathcal{S}$. This is because tensors on both Σ and $\mathcal{S}$ will occur frequently in the derivation, and the capital letters makes it easier to see which is which.

The Riemann tensor R_{abcd} on Σ is defined by

$$[\nabla_a, \nabla_b]v^c = R_{ab}{}^c{}_d v^d \tag{238}$$

where v^c is any tangent vector of Σ and $[\nabla_a, \nabla_b] = \nabla_a\nabla_b - \nabla_b\nabla_a$ is the derivative commutator. Likewise, the Riemann tensor $\tilde{R}_{ABCD}$ of $\mathcal{S}$ is given by

$$[D_A, D_B]v^C = \tilde{R}_{AB}{}^C{}_D v^D \tag{239}$$

for vectors v^D that are tangent to $\mathcal{S}$. Now, given the Riemann tensor R_{abcd} of Σ, one may ask how it relates to the Riemann tensor $\tilde{R}_{ABCD}$ of the embedded surface $\mathcal{S}$. We may answer this question by using the Gauss-Weingarten equation, which states that

$$x^a \nabla_a y^b = x^a D_a y^b - K^{ac} x_a y_c n^b \tag{240}$$

where x^a and y^b are tangent vectors of $\mathcal{S}$, K^{ac} is the second fundamental form of $\mathcal{S}$, and n^b is its normal vector field. In other words, they can be expanded in terms of a coordinate basis on $\mathcal{S}$, $x^a = x^A e^a{}_A$. The same goes for D_a and K^{ac}. This choice of notation means that we have to remember that quantities like x^a are tangential to $\mathcal{S}$, since it is not indicated by their indices.

Next, we shall consider the projection of the Riemann tensor of Σ into the tangent space of $\mathcal{S}$ by contracting it with vectors that are tangential to $\mathcal{S}$. In addition to x^a and y^a, let z^a and w^a be vectors in the tangent space of $\mathcal{S}$, and consider the quantity

$$R_{ab}{}^c{}_d x^a y^b z^d w_c = x^a y^b [\nabla_a, \nabla_b] z^c w_c = [x^a\nabla_a, y^b\nabla_b] z^c w_c - [x^a, y^a]\nabla_a z^c w_c \tag{241}$$

where $[x^a, y^a] \equiv x^b\nabla_b y^a - y^b\nabla_b x^a$. Expanding this expression gives

$$R_{ab}{}^c{}_d x^a y^b z^d w_c = \underbrace{x^a\nabla_a(y^b\nabla_b z^c)w_c}_{(1)} - \underbrace{y^b\nabla_b(x^a\nabla_a z^c)w_c}_{(2)} - \underbrace{(x^b\nabla_b y^a - y^b\nabla_b x^a)\nabla_a z^c w_c}_{(3)} \tag{242}$$

We will treat each of these three terms separately, and then recombine everything. For the first term, using the Gauss-Weingarten equation to express everything in terms of the derivative D_a gives

$$(1) = x^a\nabla_a(y^b D_b z^c - K^{bd} y_b z_d n^c)w_c = x^a\nabla_a(y^b D_b z^c)w_c - K^{bd} y_b z_d (x^a\nabla_a n^c) w_c \tag{243}$$

Where some terms vanish due to the contraction of the tangent vector w^a with the normal vector n^a. The remaining covariant derivatives ∇_a can be directly replaced by their counterpart D_a on $\mathcal{S}$ by invoking the Gauss-Weingarten equation again[33]. The result is

$$(1) = x^a D_a(y^b D_b z^c)w_c - K^{bd} y_b z_d (x^a D_a n^c) w_c \tag{244}$$

[33] Contracting equation 240 with a vector w_b that is tangential to $\mathcal{S}$ gives $w_b(x^a\nabla_a y^b) = w_b(x^a D_a y^b)$ since $w_b n^b = 0$.

The second term is identical to the first term, but with x and y interchanged:

$$(2) = y^a D_a(x^b D_b z^c)w_c - K^{bd}x_b z_d(y^a D_a n^c)w_c \tag{245}$$

Lastly, we apply the Gauss-Weingarten equation to the third term, giving

$$(3) = (x^b D_b y^a - y^b D_b x^a - \underbrace{K^{bd}x_d y_b n^a + K^{db}y_b x_d n^a}_{=0})\nabla_a z^c w_c = (x^b D_b y^a - y^b D_b x^a)\nabla_a z^c w_c \tag{246}$$

where we used the fact that the second fundamental form is symmetric. Like before, the last ∇ may be directly replaced by a D, since we contract with a tangent vector. Then

$$(3) = (x^b D_b y^a - y^b D_b x^a)D_a z^c w_c \tag{247}$$

Putting all this together, we see that

$$R_{ab}{}^c{}_d x^a y^b z^d w_c = (1) - (2) - (3) = x^a D_a(y^b D_b z^c)w_c - K^{bd}y_b z_d(x^a D_a n^c)w_c \tag{248}$$

$$- y^a D_a(x^b D_b z^c)w_c + K^{bd}x_b z_d(y^a D_a n^c)w_c - (x^b D_b y^a - y^b D_b x^a)D_a z^c w_c \tag{249}$$

$$= \underbrace{[x^a D_a, y^b D_b]w_c - [x^a, y^a]_S D_a z^c w_c}_{\bar{R}_{ab}{}^c{}_d x^a y^b z^d w_c} + K^{bd}\Big(x_b z_d \underbrace{(y^a D_a n^c)w_c}_{K^{ac}y_a w_c} - y_b z_d \underbrace{(x^a D_a n^c)w_c}_{K^{ac}x_a w_c}\Big) \tag{250}$$

Where $\tilde{R}_{ab}{}^c{}_d$ is the Riemann tensor on $\mathcal{S}$ (a quantity that lives in the tangent space of $\mathcal{S}$) in terms of coordinates on Σ. Thus, we have

$$R_{ab}{}^c{}_d x^a y^b z^d w_c = \bar{R}_{ab}{}^c{}_d x^a y^b z^d w_c + K^{bd}K^{ac}(x_b y_a - y_b x_a)z_d w_c \tag{251}$$

All indices are dummy indices, so we may rearrange them as follows:

$$R_{abcd}x^a y^b z^d w^c = \bar{R}_{abcd}x^a y^b z^d w^c + K_{bd}K_{ac}(x^b y^a - y^b x^a)z^d w^c \tag{252}$$

Renaming some indices and factorizing, this may be written as

$$R_{abcd}x^a y^b z^d w^c = (\bar{R}_{abcd} + K_{ad}K_{bc} - K_{bd}K_{ac})x^a y^b z^d w^c \tag{253}$$

We now use the fact that all of the vectors in this expression are tangent vectors of $\mathcal{S}$, which means that we can write them as a linear combination of the coordinate basis on $\mathcal{S}$: $x^a = x^i e^a{}_i$, and so on, gives

$$R_{abcd}\underbrace{x^A y^B z^C w^D}_{\text{arbitrary!}} e^a{}_A e^b{}_B e^c{}_C e^d{}_D = (\tilde{R}_{abcd} + K_{ad}K_{bc} - K_{bd}K_{ac})\underbrace{x^A y^B z^C w^D}_{\text{arbitrary!}} e^a{}_A e^b{}_B e^c{}_C e^d{}_D \tag{254}$$

The linear coefficients that specify the tangent vectors are completely arbitrary, so the equality must hold without them:

$$R_{abcd}e^a{}_A e^b{}_B e^c{}_C e^d{}_D = (\tilde{R}_{abcd} + K_{ad}K_{bc} - K_{bd}K_{ac})e^a{}_A e^b{}_B e^c{}_C e^d{}_D \tag{255}$$

All quantities on the right hand side of this equation live in the tangent space of $\mathcal{S}$, so we can write

$$R_{abcd}e^a{}_A e^b{}_B e^c{}_C e^d{}_D = (\tilde{R}_{ijkl} + K_{il}K_{jk} - K_{jl}K_{ik}) \tag{256}$$

We can almost see the finish line now. Contracting this equation with the induced metric γ^{ij} on $\mathcal{S}$, we find that

$$R_{abcd}\underbrace{e^a{}_A e^c{}_C \gamma^{AC}}_{=\gamma^{ac}}\underbrace{e^b{}_B e^d{}_D \gamma^{BD}}_{\gamma^{bd}} = R_{\mathcal{S}} + K_{AB}K^{AB} - p^2 \tag{257}$$

where $R_{\mathcal{S}}$ is the Riemann scalar (intrinsic curvature) of $\mathcal{S}$, and p is the mean extrinsic curvature of $\mathcal{S}$. Now, we use the fact that

$$\gamma^{ac} = g^{ac} - n^a n^c \tag{258}$$

where n^a is the normal vector of $\mathcal{S}$ and γ^{ac} is the metric on $\mathcal{S}$ expressed in terms of coordinates on Σ. The left hand side of equation 257 then becomes

$$\begin{aligned}
R_{abcd}\gamma^{ac}\gamma^{bd} &= R_{abcd}(g^{ac} - n^a n^c)(g^{bd} - n^b n^d) = \\
&= \underbrace{R_{abcd}g^{ac}g^{bd}}_{=R_\Sigma} + \underbrace{R_{abcd}n^a n^b n^c n^d}_{=0} - \underbrace{R_{abcd}n^a n^c g^{bd}}_{=R_{ac}n^a n^c} - \underbrace{R_{abcd}n^b n^d g^{ac}}_{=R_{bd}n^b n^d} \\
&= R_\Sigma - 2R_{ac}n^a n^c = -2G_{ac}n^a n^c
\end{aligned} \tag{259}$$

where G_{ac} is the Einstein tensor. Finally, we have arrived at the remarkable theorem:

$$2G_{ac}n^a n^c = p^2 - R_\mathcal{S} - K_{AB}K^{AB} \tag{260}$$

Specifically, if we choose $\mathcal{S}$ to be a three-dimensional spacelike hypersurface and Σ to be four-dimensional spacetime, then n^a must be a timelike unit vector. The Einstein field equations then gives us a simple way to express the left-hand side of the above equation:

$$2G_{ac}n^a n^c = \underbrace{16\pi T_{ac}n^a n^c}_{2\mu} - 2\Lambda \underbrace{g_{ac}n^a n^c}_{-1} = 2\mu + 2\Lambda \tag{261}$$

where μ is the energy density of Σ and Λ is the cosmological constant. This gives

$$R_\mathcal{S} = 2\mu + 2\Lambda + K_{AB}K^{AB} - p^2 \tag{262}$$

10.2 Second variation of the area

As shown in the preliminary section, the mean curvature p can be interpreted as the change of the area of a surface if it was transported outwards along the normals. The directional derivative (or Lie derivative) of p along the normal direction is therefore the second variation of the area. The second variation of the area theorem gives an explicit expression for this derivative, which has a very simple statement in terms of the intrinsic and extrinsic curvatures of the surface and the manifold it is embedded in. Proving this theorem is notoriously complicated; the reader is encouraged to take a look in for example Spivak, volume four [28], where the proof is over thirty pages long.

Since this theorem is so central to the proof of the monotonicity property of the Geroch mass, and indeed important in general when considering geometric flows of surfaces, we shall provide a derivation of it here. The mathematical details will not be covered as extensively as Spivak or other authors gives them, so that it is more of a derivation than a proof. The calculation is very dense, but necessarily so.

While it is possible to state the theorem more generally, we shall only be bothered with the specific case which is of use to us; namely two-dimensional surfaces embedded in three-dimensional hypersurfaces.

Let Σ be a three-dimensional spatial hypersurface, coordinatized by $x^a = (x^1, x^2, x^3)$, with metric tensor g_{ab} and covariant derivative ∇_a. Let $\mathcal{S}$ be a two-dimensional surface embedded within Σ, coordinatized by $u^A = (u^1, u^2)$, with induced metric γ_{AB} (inherited from Σ), its associated covariant derivative D_A, and second fundamental form K_{AB}. We are sticking to the convention to use captial indices for quantities in the tangent space of $\mathcal{S}$, since a lot of tensors in both tangent spaces will occur.

Let the vector field $\vec{n}$ be normal to the surface, and let an overdot denote Lie differentiation with respect to a normal flow:

$$\dot{p} = \mathcal{L}_{\phi\vec{n}}\, p \tag{263}$$

where ϕ is some scalar function that determines the speed of the flow. Then, the second variation of the area is given by

$$\dot{p} = -D^A D_A \phi - \frac{1}{2\phi}\left(p^2 - K^{AB}K_{AB} + R_S - R_\Sigma\right) \tag{264}$$

Preliminaries

To begin with, we shall state some necessary identities and relations. The tangent vectors of $\mathcal{S}$ are given by

$$e^a{}_A = \frac{\partial x^a}{\partial u^A} \tag{265}$$

The induced metric on $\mathcal{S}$ is given by

$$\gamma_{AB} = \vec{e}_A \cdot \vec{e}_B = g_{ab} e^a{}_A e^b{}_B \tag{266}$$

The second fundamental form of $\mathcal{S}$ is given by

$$K_{AB} = -\vec{n} \cdot \nabla_{\vec{e}_A} \vec{e}_B = \vec{e}_B \nabla_{\vec{e}_A} \vec{n} = -n_b e^a{}_A \nabla_a e^b{}_B = e^a{}_A e^b{}_B \nabla_a n_b, \quad K_{AB} = K_{BA} \tag{267}$$

We also note that

$$\mathcal{L}_{\vec{n}} g_{ab} = n^c \nabla_c g_{ab} + g_{ac} \nabla_b n^c + g_{bc} \nabla_a n^c = \nabla_b n_a + \nabla_a n_b \tag{268}$$

Which gives us another expression for the second fundamental form:

$$e^a{}_A e^b{}_B \mathcal{L}_{\vec{n}} g_{ab} = e^a{}_A e^b{}_B (\nabla_b n_a + \nabla_a n_b) = K_{BA} + K_{AB} = 2K_{AB} \iff K_{AB} = \frac{1}{2} e^a{}_A e^b{}_B \mathcal{L}_{\vec{n}} g_{ab} \tag{269}$$

The Gauss-Weingarten equation:

$$\nabla_{\vec{e}_A} \vec{e}_B = D_{\vec{e}_A} \vec{e}_B - K_{AB} \vec{n} \tag{270}$$

where D is the covariant derivative on $\mathcal{S}$, such that $D_{\vec{e}_A}\vec{e}_B = \Gamma^C{}_{AB}\vec{e}_C$. We will also need the derivative of the metric on $\mathcal{S}$, which is

$$\dot{\gamma}_{AB} = \mathcal{L}_{\phi\vec{n}}\gamma_{AB} = \mathcal{L}_{\phi\vec{n}}(e^a{}_A e^b{}_B g_{ab}) = e^a{}_A e^b{}_B \mathcal{L}_{\phi\vec{n}} g_{ab} \tag{271}$$

It may be assumed that $\mathcal{L}_{\phi\vec{n}}\vec{e}_A = 0$, which means that the coordinatization of the surface does not change as it evolves (this is not necessary always the case, but the coordinates can be chosen in such a way that this is true). This means that

$$\begin{aligned}\dot{\gamma}_{AB} &= e^a{}_A e^b{}_B \mathcal{L}_{\phi\vec{n}} g_{ab} = e^a{}_A e^b{}_B \big(\phi n^c \nabla_c g_{ab} + \nabla_a(\phi n_b) + \nabla_b(\phi n_a)\big) \\ &= \phi e^a{}_A e^b{}_B (n^c \nabla_c g_{ab} + \nabla_a n_b + \nabla_b n_a) = \phi e^a{}_A e^b{}_B \mathcal{L}_{\vec{n}} g_{ab} = 2\phi K_{AB}\end{aligned} \tag{272}$$

and, since $\partial(\gamma_{AB}\gamma^{BC}) = 0$, it also follows that $\dot{\gamma}^{AB} = -2\phi K^{AB}$.

Derivation

We are now in a position to derive the second variation of the area. From the definition $p = \gamma^{AB}K_{AB}$, we have

$$\dot{p} = \gamma^{AB}\dot{K}_{AB} + \dot{\gamma}^{AB}K_{AB} \tag{273}$$

We will need to work out what $\dot{K}_{AB}$ is, since we know the rest already. We have

$$\dot{K}_{AB} = \mathcal{L}_{\phi\vec{n}}K_{AB} = \mathcal{L}_{\phi\vec{n}}(e^a{}_A e^b{}_B \nabla_b n_a) = e^a{}_A e^b{}_B \mathcal{L}_{\phi\vec{n}}(\nabla_b n_a) \tag{274}$$

using the assumption that $\mathcal{L}_{\phi\vec{n}} e^a{}_A = 0$. Continuing, we have

$$\mathcal{L}_{\phi\vec{n}}(\nabla_b n_a) = \phi n^c \nabla_c(\nabla_b n_a) + \nabla_c n_a \nabla_b(\phi n^c) + \nabla_b n_c \nabla_a(\phi n^c) \tag{275}$$

Now, recall the definition of the Riemann tensor:

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)\xi_c = R_{abcd}\xi^d \tag{276}$$

where $\vec{\xi}$ is an arbitrary vector. This implies that

$$\nabla_c\nabla_b n_a = R_{cbad}n^d + \nabla_b\nabla_c n_a \tag{277}$$

Using this, we find that

$$\mathcal{L}_{\phi\vec{n}}(\nabla_b n_a) = \underbrace{\phi R_{cbad}n^c n^d}_{-\phi R_{acbd}n^c n^d} + \underbrace{\phi n^c \nabla_b\nabla_c n_a + \nabla_c n_a \nabla_b(\phi n^c)}_{\nabla_b(\phi n^c \nabla_c n_a)} + \nabla_b n_c \nabla_a(\phi n^c) \tag{278}$$

Thus:

$$\dot{K}_{AB} = \underbrace{-\phi R_{acbd} e^a{}_A n^c e^b{}_B n^d}_{(3)} + \underbrace{e^a{}_A e^b{}_B \nabla_b(\phi n^c \nabla_c n_a)}_{(2)} + \underbrace{e^a{}_A e^b{}_B \nabla_b n_c \nabla_a(\phi n^c)}_{(1)} \tag{279}$$

Where we have labeled each term by a number so that we may treat them separately. The goal is now to simplify each term as far as possible. Starting with (1), we have

$$(1) = (e^b{}_B \nabla_b n_c)\big(e^a{}_A \nabla_a(\phi n^c)\big) = \phi(e^b{}_B \nabla_b n_c)(e^a{}_A \nabla_a n^c) \tag{280}$$

where it's possible to factor out the ϕ due to the fact that $n^c\nabla_b n_c = 0$. Recall that $K_{BC} = e^c{}_C(e^b{}_B \nabla_b n_c)$. This gives

$$K_B{}^C e_{dC} = \underbrace{e_{dC}e^{cC}}_{=\delta_d{}^c - n_d n^c} (e^b{}_B \nabla_b n_c) \quad \Longrightarrow \quad K_B{}^C e_{cC} = e^b{}_B \nabla_b n_c \tag{281}$$

and by raising and renaming indices, we get

$$K_A{}^D e^c{}_D = e^a{}_A \nabla_a n^c \tag{282}$$

putting this into (1), we get

$$(1) = \phi K_B{}^C K_A{}^D \underbrace{e_{cC}e^c{}_D}_{\gamma_{CD}} = \phi K_B{}^C K_{AC} \tag{283}$$

Moving on to the next term, we have

$$(2) = e^a{}_A e^b{}_B \nabla_b(\phi \underline{n^c \nabla_c n_a}) \tag{284}$$

We will begin by looking at the underlined part, with the last index raised.

$$n^c \nabla_c n^a = g^{da} n^c \nabla_c n_d \tag{285}$$

Now, using the fact that

$$g^{da} = \gamma^{AB} e^a{}_A e^d{}_B + n^a n^d \tag{286}$$

we get

$$n^c \nabla_c n^a = \gamma^{AB} e^a{}_A e^d{}_B n^c \nabla_c n_d + \underbrace{n^d n^a n^c \nabla_c n_d}_{=0} = \gamma^{AB} e^a{}_A n^c \underbrace{(e^d{}_B \nabla_c n_d)}_{=-n_d \nabla_c e^d{}_B} = -\gamma^{AB} e^a{}_A n_d \underset{*}{\underline{n^c \nabla_c e^d{}_B}} \tag{287}$$

since $0 = \nabla_c(n_d e^d{}_B) = n_d \nabla_c e^d{}_B + e^d{}_B \nabla_c n_d$. Now, we will treat the second underlined part. Recall that $\mathcal{L}_{\phi\vec{n}} \vec{e}_A = 0$. This gives us:

$$0 = \phi n^c \nabla_c e^d{}_B - e^c{}_B \nabla_c(\phi n^d) = \phi n^c \nabla_c e^d{}_B - \phi \underbrace{e^c{}_B \nabla_c n^d}_{K_B{}^D e^d{}_D} - n^d \underbrace{e^c{}_B \nabla_c \phi}_{D_B \phi} \tag{288}$$

where D is the intrinsic derivative on $\mathcal{S}$. We thus have

$$\underset{*}{\underline{n^c \nabla_c e^d{}_B}} = K_B{}^D e^d{}_D + \frac{1}{\phi} n^d D_B \phi \tag{289}$$

Substituting back, we have

$$\underline{n^c \nabla_c n^a} = -\gamma^{AB} e^a{}_A K_B{}^D \underbrace{n_d e^d{}_D}_{=0} - \frac{1}{\phi} \gamma^{AB} e^a{}_A \underbrace{n_d n^d}_{=1} D_B \phi = -\frac{1}{\phi} e^a{}_A D^A \phi \tag{290}$$

Lowering the index a and substituting into (2), we get

$$(2) = -e^a{}_A e^b{}_B \nabla_b(e_{aC} D^C \phi) = -e^b{}_B \nabla_b(\underbrace{e^a{}_A e_{aC}}_{\gamma_{AC}} D^C \phi) + e^b{}_B e_{aC} D^C \phi \nabla_b e^a{}_A \tag{291}$$

$$= -e^b{}_B \nabla_b(D_A \phi) + e_{aC} D^C \phi \underline{e^b{}_B \nabla_b e^a{}_A} \tag{292}$$

The Gauss-Weingarten equation tells us that this underlined quantity is:

$$\underline{e^b{}_B \nabla_b e^a{}_A} = e^b{}_B D_b e^a{}_A - K_{AB} n^a \tag{293}$$

Thus:

$$(2) = -e^b{}_B \nabla_b(D_A \phi) + (e_{aC} D^C \phi) \underbrace{(e^b{}_B D_b e^a{}_A)}_{=\Gamma^D{}_{AB} e^a{}_D} - D^C \phi K_{AB} \underbrace{e_{aC} n^a}_{=0} \tag{294}$$

Using the fact that $D_{\vec{e}_A} \vec{e}_B = \Gamma^D{}_{AB} \vec{e}_D$. The ∇_b in the first term acts on a quantity defined on $\mathcal{S}$, and is then projected down to $\mathcal{S}$, so we can replace it by a regular derivative ∂_B on $\mathcal{S}$. Hence:

$$(2) = -\partial_B D_A \phi + \underbrace{e_{aC} e^a{}_D}_{=\gamma_{CD}} D^C \phi \Gamma^D{}_{AB} = -\left(\partial_B D_A \phi - \Gamma^D{}_{AB} D_D \phi\right) = -D_B D_A \phi \tag{295}$$

Moving on to the third term,

$$(3) = -\phi R_{acbd} e^a{}_A n^c e^b{}_B n^d \tag{296}$$

This can be rewritten using one of the Gauss-Codazzi equations[34], which tells us that

$$R_{acbd}e^a{}_A n^c e^b{}_B n^d = R_{ac}e^a{}_A e^c{}_B - R_{AB} - K_{AD}K^D{}_B + pK_{AB} \tag{297}$$

Thus

$$(3) = -\phi\left(R_{ac}e^a{}_A e^c{}_B - R_{AB} - K_{AD}K^D{}_B + pK_{AB}\right) \tag{298}$$

Finally, putting everything together, we have

$$\dot{K}_{AB} = (3) + (2) + (1) = -D_B D_A\phi - \phi\left(R_{ac}e^a{}_A e^c{}_B - R_{AB} - 2K_{AD}K^D{}_B + pK_{AB}\right) \tag{299}$$

and

$$\begin{aligned}
\dot{p} &= \gamma^{AB}\dot{K}_{AB} + \underbrace{\dot{\gamma}^{AB}}_{-2\phi K^{AB}} K_{AB} = \\
&= -D^A D_A\phi - \phi\big(R_{ac}\underbrace{\gamma^{AB}e^a{}_A e^c{}_B}_{g^{ac}-n^a n^c} - \underbrace{\gamma^{AB}R_{AB}}_{=R_S} - 2\underbrace{\gamma^{AB}K_{AD}K^D{}_B}_{=K_{AB}K^{AB}} + p\underbrace{\gamma^{AB}K_{AB}}_{=p}\big) - 2\phi K^{AB}K_{AB} \\
&= -D^A D_A\phi - \phi\big(R_\Sigma - R_{ac}n^a n^c - R_S + p^2\big)
\end{aligned} \tag{300}$$

From the Theorema Egregium, we have

$$R_{ab}n^a n^b = \frac{1}{2}(R_\Sigma - R_S + p^2 - K_{AB}K^{AB}) \tag{301}$$

Which, inserted in the above, gives us

$$\dot{p} = -D^A D_A\phi - \frac{1}{2\phi}\left(R_\Sigma + R_S - K_{AB}K^{AB} - \phi p^2\right) \tag{302}$$

[34]Note the similarity to the Theorema Egregium, which we derived above. It is derived in a similar fashion.

11 Appendix C - Algorithm

11.1 The main program

```
close all
clear all

%%%%%%%%%%%%%%%%%%%%%%%%

% Program parameters:
N = 200;                    % Number of data points
targetTime = 3;             % How long to evolve the surface
k = 1/24;                   % Plot every k:th second (approximately)
smoothfirstderivative = 1;  % Whether or not to apply smoothing to first derivative
                            % if yes, algorithm runs much faster.
smoothingparameter = 0.999; % Used to remove noise

% Spheroid parameters
a = 1;                      % Spheroid lateral axis
b = 2;                      % Spheroid vertical axis (axis of symmetry)

%Plots:
plot2d = 1;                 % Plot the main thing
plotall = 1;                % Plots derivatives and curvatures too
plot3d = 0;                 % Make 3D plots of the surface
record = 0;                 % Record each step into a movie (3D plots). Movie is
                            % saved to working folder as 'imcf.avi'.
                            % warning: 3 seconds ~= 100mb with 640x480 resolution.

plotaxis3d = 12;            % When plotting in 3d; plots x,y,z between
                            % +/- plotaxis3d.
xresolution = 640;          % Resolution for recording.
yresolution = 480;          % Resolution for recording.

% Don't change these:
DeltaX = pi/N;              % Spatial resolution is determined by number of points
currentTime = 0;            % Keeps track of time
plotTimer = 0;              % Counts time since last plot
j = 1;                      % Counts number of loops

%%%%%%%%%%%%%%%%%%%%%%%%

%%%% Program starts here %%%%

x = linspace(0, pi, N);       % Create a vector of x-values.
x1 = x(1:N-1)-pi;             % Create a vector of x-values extended one period
x2 = x(2:N)+pi;               % in both directions.
xtended = [x1'; x'; x2'];

% Create arrays for Phi (the solution) and its derivatives
Phi = zeros(N, 2);
DPhix = zeros(N,1);
DDPhixx = zeros(N,1);

% Define the initial surface
r = (sin(x).^2/a^2 + cos(x).^2/b^2).^(-1/2);

% Create array for K (curvature) values
```

```matlab
K = zeros(N,1);

% Set first column of Phi to be the initial surface
Phi(1:N, 1) = r;

% Plot the original curve (as closed contour in the plane)
if(plot2d)
    figure(1)
    hold on
    polar([x'+pi/2; -flipud(x(2:N)')+pi/2; x(1)+pi/2], ...
          [Phi(1:N, 1); flipud(Phi(2:N,1)); Phi(1,1)]);
end

% Plot the original curve (radius as function of angle)
if(plotall)
    figure(5)
    hold on
    plot(x', Phi(1:N, 1))
end

% The following part is used for recording 3d-plots into a movie.
if(record)
    frames = floor(1/k * targetTime);
    mov(frames) = struct('cdata',[],'colormap',[]);
    frame = 1;
    fig = figure(9);
    set(fig, 'Units', 'points', 'Position', [0,0,xresolution,yresolution]);
end

%%%%%%%%%%%%%%%%% Main loop %%%%%%%%%%%%%%%%%

while currentTime < targetTime

    % Given Phi at currentTime, we calculate Phi at currentTime+DeltaT.

    % Calculate discretized x-derivative
    y = NCMirrorDiff(Phi(1:N,1), DeltaX, 0); % NCMirrorDiff is a function that
                                             % numerically differentiates a function,
                                             % assuming that it is periodic and
                                             % mirror symmetric around the origin.
    if(smoothfirstderivative)
        y1 = -flipud(y(2:N));
        y2 = -flipud(y(1:N-1));
        padded = [y1; y; y2];
        temp = fit(xtended, [y1; y; y2], 'smoothingspline', ...
                   'SmoothingParam', smoothingparameter);
        DPhix(1:N) = temp(x);
    else
        DPhix(1:N) = y;
    end
    % Calculate discretized xx-derivative
    DDPhixx(1:N) = NCMirrorDiff(DPhix, DeltaX, 1);

    % Calculate curvature:
    % The extrinsic curvature of a rotationally symmetric body is the sum
    % of the curvature of the two-dimensional curve that was rotated and an
    % additional term representing the extra curvature in the third dimension.
    K2D = Phi(1:N, 1) .* (Phi(1:N, 1).^2 + DPhix(1:N).^2).^(-3/2) ...
          .* ((Phi(1:N, 1)-DDPhixx(1:N)) + 2./Phi(1:N, 1) .* DPhix(1:N).^2);
    Kextra = (Phi(1:N, 1).^2 + DPhix(1:N).^2).^(-1/2) ...
```

```
          .* (1 - DPhix(1:N)./Phi(1:N, 1) .* cot(x'));
Kextra(1) = Kextra(2);  % The cotangent becomes problematic at the end-points,
Kextra(N) = Kextra(N-1);% so we replace them with their neighbours. This
K = K2D + Kextra;       % approximation is negligible as long as N is large.

% Determine time-step. E3 is the derivative of F (the right hand side)
% w.r.t. the second derivative of phi.
E3 = -Phi(1:N,1).^2.*(DPhix.^2 + Phi(1:N,1).^2).^2 ./ (Phi(1:N,1).* ...
     (-3.*DPhix.^2 + (DDPhixx-2.*Phi(1:N,1)).*Phi(1:N,1)) ...
     + DPhix.*(DPhix.^2 + Phi(1:N,1).^2).*cot(x)').^2;
E3(1) = E3(2);
E3(end) = E3(end-1);
if(smoothfirstderivative)
    DeltaT = DeltaX / max([2*max(abs(E3)), 1]);
else
    DeltaT = DeltaX^2 / max([4*max(abs(E3)), 1]);
end

% Calculate phi at next point in time
F = -(1./K).*(1+DPhix.^2./Phi(1:N, 1).^2).^(1/2);
Phi(1:N,2) = Phi(1:N,1) - DeltaT .* F;

% Draw plots
if(plotTimer >= k)
    plotTimer = 0;
    if(plotall) % Auxilliary plots
        figure(2)
        plot(x, DPhix(1:N));
        title('First Derivative')
        hold on
        figure(3)
        plot(x,DDPhixx(1:N));
        title('Second derivative')
        hold on
        figure(60)
        hold on
        plot(x, E3);
        title('Derivative of right hand side w.r.t. d^2 phi/dx^2')
        figure(8)
        plot(x,Kextra)
        hold on
        title('Extra Curvature')
        figure(4)
        plot(x,K)
        title('Curvature')
        hold on
        figure(6)
        plot(x, 1./K);
        title('Inverse curvature')
        hold on
        figure(7)
        plot(x, F);
        title('Right hand side')
        hold on;
        figure(5)
        plot(x', Phi(1:N, 2))
        title('r(theta)')
        hold on
    end
    if(plot2d) % The main plot
```

```
            figure(1)
            polar([x'+pi/2; -flipud(x(2:N)')+pi/2; x(1)+pi/2], ...
                  [Phi(1:N, 2); flipud(Phi(2:N,2)); Phi(1,2)]);
            axis equal
        end
        if(plot3d)
            fig = figure(9);
            DrawSurfaceOfRevolution(x, Phi(1:N,2), N)
            axis([-plotaxis3d plotaxis3d -plotaxis3d ...
                   plotaxis3d -plotaxis3d plotaxis3d])
            if(record)
                mov(frame) = getframe(fig);
                frame = frame + 1;
            end
        end
    end

    % Calculate the integral of the curvature squared over the surface.
    % This should decrease as the surface evolves.
    Q(j) = 2*pi*sum(K.^2 .* Phi(1:N,1).^2 .* sin(x)' .* DeltaX);

    % Set the calculated Phi to current Phi, begin again
    Phi(1:N,1) = Phi(1:N, 2);

    j = j + 1;
    currentTime = currentTime + DeltaT;
    plotTimer = plotTimer + DeltaT;
end

if(plotall)
    figure(10)
    plot(Q)
    title('Integral of curvature squared')
end

if(record)
    for i = 1:frames
       if(length(mov(i).cdata) == 0)
           break;
       end
    end
    mov2 = mov(1:(i-1));
    movie2avi(mov2, 'imcf.avi');
end
```

11.2 Help functions

```
function Df = NCMirrorDiff(f, Delta, even)
    N = length(f);

    fNext(1:N-1) = f(2:N);
    fNext(N) = f(N-1);
    fPrev(2:N) = f(1:N-1);
    fPrev(1) = f(2);
    Df(1:N,1) = (fNext(1:N) - fPrev(1:N))./(2*Delta);

    if(even)
        Df(1,1) = (fNext(1) + fPrev(1))./(2*Delta);
        Df(N,1) = (-fNext(N) - fPrev(N))./(2*Delta);
```

```
    end

end


function DrawSurfaceOfRevolution(theta, r, N)

clear tri
clear X
clear Y
clear Z

X = zeros(N*N, 1);
Z = zeros(N*N, 1);
Y = zeros(N*N, 1);

for i = 1:N
    for j = 1:N
        [X(N*(i-1) + j), Y(N*(i-1) + j), Z(N*(i-1) + j)] = ...
            pol2cart(2*pi/N *(j-1),r(i).*sin(theta(i)),r(i).*cos(theta(i)));
    end
end

tri = zeros(2*(N^2-N), 3);

for i = 1:(N^2-N)

    polygon = 2*i - 1;

    if(mod(i,N) == 0)
        tri(polygon,1) = i;
        tri(polygon,2) = i + N;
        tri(polygon,3) = i + 1;

        tri(polygon+1,1) = i;
        tri(polygon+1,2) = i + 1;
        tri(polygon+1,3) = i - N + 1;
    else
        tri(polygon,1) = i;
        tri(polygon,2) = i + N;
        tri(polygon,3) = i + N + 1;

        tri(polygon+1,1) = i;
        tri(polygon+1,2) = i + N + 1;
        tri(polygon+1,3) = i + 1;
    end


end

h = trisurf(tri, X, Y, Z);

view(33,15)
shading interp
lighting phong
light
colormap summer

end
```