Squashing anti-de Sitter space

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Abstract

A space-time analogue to the well known Berger sphere, a class of geometries that interpolate between 3-dimensional and 2-dimensional anti-de Sitter space, is studied. Special attention is paid to the conformal boundary of the space as, for 3-dimensional anti-de Sitter space, it is in itself locally conformal to 2-dimensional anti-de Sitter space; and in this light the squashing of anti-de Sitter space could be viewed as a kind of interpolation of the space onto its own boundary. It is found that squashed anti-de Sitter space does not posses a conformal boundary at all. Also it is investigated if the space-times are solutions to a conformal generalization of the Einstein equation, called the Einstein-Weyl equation. The answer is found to be - not quite, but almost.

Acknowledgements

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Contents

Chapter 1

Squashing the 3-sphere

The 3-sphere (S^3) is in many respects like the ordinary sphere, only three-dimensional; a totally symmetric space with constant positive curvature. It can be considered as the surface

$$X^2 + Y^2 + Z^2 + W^2 = 1\,, \tag{1.1}$$

in a four dimensional Euclidean space with Cartesian coordinates (X,Y,Z,W). The symmetries of S^3 are described by the group of rotations on the four-dimensional embedding space, $O(4)$. When trying to visualize the 3-sphere, describing it as a surface in a four-dimensional space is no good because four-dimensional space is notoriously hard to imagine. Instead one usually projects it onto flat 3-space and gets a somewhat distorted, but manageable picture. A mapping appropriate to use here is the stereographic projection, which looks like this:

$$\begin{aligned} X &= \frac{2x}{1+r^2}\,, \qquad & Y &= \frac{2y}{1+r^2}\,, \\ Z &= \frac{2z}{1+r^2}\,, \qquad & W &= \frac{1-r^2}{1+r^2}\,, \end{aligned} \tag{1.2}$$

$$r^2 = x^2 + y^2 + z^2\,. \tag{1.3}$$

The stereographic projection is a conformal map which means that the metric of S^3 is conformal to the flat metric, that is, it is the flat metric multiplied by a factor that is a function of the coordinates:

$$ds^2_{S^3} = \frac{4}{(1+r^2)}(dx^2 + dy^2 + dz^2)\,. \tag{1.4}$$

The coordinates (x,y,z) are called stereographic coordinates for the obvious reason. It is of course possible to map the embedded surface to flat space in different ways and obtain different coordinate systems useful for various things, but this is not something I will consider here. Stereographic coordinates are useful because they give a true picture of angles and small shapes; distances are distorted but angles are conserved by the mapping. Another interesting property of conformal maps is that they can give a picture of how a space looks "infinitely far away". $\mathbb{R}^3$ is an infinite non-compact space while S^3 is a compact space. The stereographic projection considered above maps the north pole of S^3 to infinity of $\mathbb{R}^3$. It turns out that *any* conformal map of the 3-sphere to flat space can take at most one point to infinity; or the converse: *A conformal map of flat space will project infinity to a point.* This will allow us to say, that in a way, infinity of flat Euclidean space *is* a point. For a more thorough, and very readable, description of conformal maps see [5]. A more interesting example of infinity will appear in the next chapter when we are looking at anti-de Sitter space.

We would like to know how the geodesics and the symmetry flow lines look in this picture. Geodesics appear as lines through the origin, circles intersecting the equator (the unit-sphere) at antipodal points, or great circles on the equator. The symmetry flow is generated by the Killing vector fields, which in the case of the 3-sphere are the generators of rotation:

$$J_{ij} = X_i \partial_j - X_j \partial_i \,. \tag{1.5}$$

There are 6 such linearly independent vectors, and their flow lines are all intrinsically circles. In our picture some of them look a bit different because the W coordinate has been singled out by the stereographic projection. Consider first rotations in the XY-plane:

$$J_{XY} = X\partial_Y - Y\partial_X = x\partial_y - y\partial_x \,. \tag{1.6}$$

These still look like rotations in the stereographic picture and their flow lines are circles. The z-axis is fixed under this rotation. If we instead rotate in the ZW-plane we have fixed points where

$$Z = W = 0 \qquad \Leftrightarrow \qquad r = 1,\, z = 0\,. \tag{1.7}$$

This is the unit circle in the xy-plane. The flow lines of rotations are still circles around this line of fixed points; in the stereographic picture this looks like circles on a torus.

An interesting and important property of the 3-sphere is that there exists Killing vector fields that do not have any fixed points, for example the field

$$\xi = J_{XY} + J_{ZW} = X\partial_Y - Y\partial_X + Z\partial_W - W\partial_Z \,. \tag{1.8}$$

This vector field has flow lines that, like those of the J_{ZW}, lie on tori around the unit circle r=1; but they now also wind one time around the z-axis like those of the J_{XY} (see fig 1.1). Since no points in the space are fixed under the action of this vector field it is clear that every point must lie on a flow line of ξ; *the flow lines of ξ fill the whole of S^3*. This is very different from the 2-sphere, where it is impossible to find a Killing vector field without fixed points.

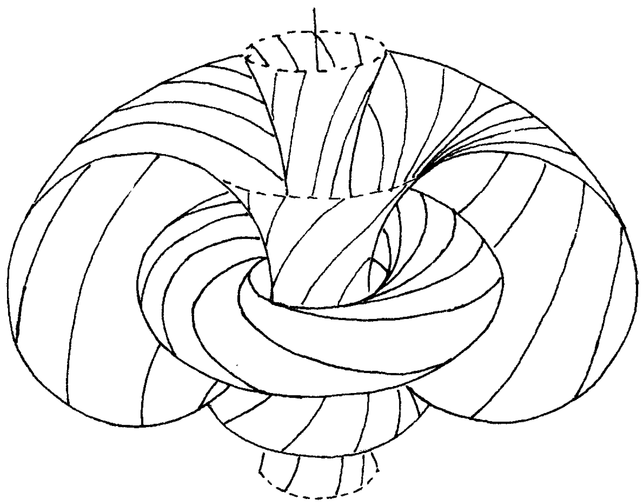

Figure 1.1: Clifford parallels on S^3. Picture copied from Penrose

Now, the geometry of S^3 is such that two initially parallel geodesics will eventually intersect at some point, this because of the constant positive curvature (think of great circles on the sphere). It is however possible to define another meaning of "parallel" here. Instead of looking at two lines whose tangent vectors are parallel displaced to each other, we rotate one of the lines such that the two geodesics become skew to each other. Now when we move along the geodesics the distance will increase between them because they are pointing in different directions. It is possible to adjust the skewness in such a way that the diverging effect of the skewness exactly cancels the converging effect of the geometry and the lines stay at a constant distance of each other along the whole of their length. Such lines are called Clifford parallels. For S^3 it is possible to completely foliate the space with Clifford parallels [1], and in fact, this foliation is exactly the space-filling family of flow-lines of ξ constructed above.

This construction is what is known as the Hopf fibration of the 3-sphere, and is a prime example of a *fibre bundle*, a concept of great interest in several areas of physics. Since the distance between two Clifford parallels is independent of where one is along them, one can talk about distance between the Clifford parallels as whole objects. We define the distance between two Clifford parallels as the shortest distance between any two points, one on each line. Looking at S^3 in this way it becomes a space of circles (a space whose points are circles). It turns out that, actually, this space of circles is a 2-sphere. *The 3-sphere is a 2-sphere of circles.* In a construction like this the 2-sphere is sometimes called *the quotient space of* S^3 *with* S^1, denoted as $S^2 = S^3/S^1$; the Clifford parallels are in a sense "factored out of the space" and what is left is the 2-sphere.

The Berger spheres is a family of geometries that interpolate between the 3-sphere and the 2-sphere; instead of just factoring away the Clifford parallels we construct a continuous transition between S^3 and S^2. It is constructed by gradually decreasing the distance along the Clifford parallels down to zero. Being more explicit we can look at the metric of the 3-sphere expressed in Euler coordinates,

$$ds^2 = \frac{1}{4}(d\tau^2 + d\theta^2 + d\phi^2 + 2\cos\theta d\tau d\phi) = \tag{1.9}$$

$$= \frac{1}{4}((d\tau + \cos\theta\, d\phi)^2 + d\theta^2 + \sin^2\theta d\phi^2)\,. \tag{1.10}$$

where τ here is the coordinate along the Clifford parallels. The squashed 3-sphere is obtained by inserting a new parameter λ in the metric with values ranging between 1 and 0.

$$ds^2 = \frac{1}{4}(\lambda^2\,(d\tau + \cos\theta\, d\phi)^2 + d\theta^2 + \sin^2\theta d\phi^2)\,. \tag{1.11}$$

For $\lambda = 1$ the metric is that of the 3-sphere, and for $\lambda = 0$ it is the metric of the 2-sphere (if one ignores the factor of $\frac{1}{4}$ in front). For any value in between it is the metric of a Berger sphere; topologically still a 3-sphere but distorted and less symmetric. This construction is well known and described in more detail elsewhere. It is my intention with this thesis to describe another, similar, construction; a family of geometries that interpolate between 3-dimensional and 2-dimensional anti-de Sitter space.

Chapter 2

Anti-de Sitter space

Anti-de Sitter space is a space-time of constant negative curvature and a solution to Einsteins equations with negative cosmological constant. It is presently believed that the universe possesses a positive cosmological constant, which at first sight makes the study of anti-de Sitter space more of a curiosity than relevant physics. Nevertheless, as one of few simple solutions of Einsteins equations it merits some attention, and is covered, for example, in Hawking's and Ellis's *The large scale structure of space-time.* String theorists have given anti-de Sitter space more than just some attention, mainly because of the properties of its conformal boundary which makes it possible to define conformal field theories on the boundary that correspond to string theories in the interior. More on the conformal boundary of anti-de Sitter space below. In this chapter we will take a detailed look at anti-de Sitter space, especially from a viewpoint that later will enable us to understand the squashed space.

2.1 Hyperbolic space

The understanding of anti-de Sitter space will greatly be helped by some familiarity with hyperbolic space, thus this is where we will begin. Hyperbolic space, or in two dimensions the hyperbolic plane, H^2, is a space of constant negative curvature analogous to the sphere. Like the sphere it can be considered as the set of points at an equal distance from the origin in a higher dimensional flat space. Unlike the sphere this flat space has indefinite signature metric,

$$ds^2 = -dT^2 + dX^2 + dY^2 \,. \tag{2.1}$$

The hyperbolic plane is usually defined as the upper sheet of the hyperboloid

$$-T^2 + X^2 + Y^2 = -1 \,. \tag{2.2}$$

One could have considered the surface (2.2) embedded in ordinary space, but this would not have been a space of constant curvature. The necessity of using space-time as embedding space comes from the fact that both the quadric (2.2) and Minkowski space-time are invariant under the pseudo-orthogonal transformations $SO(1,2)$ while Euclidean space and the sphere has rotational symmetry

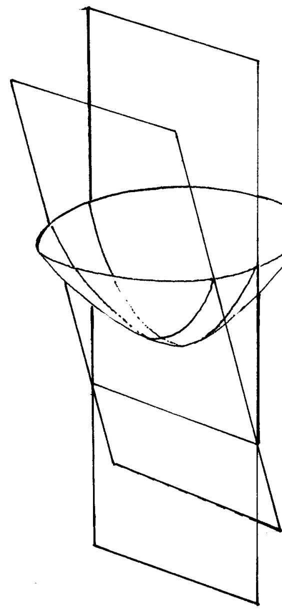

Figure 2.1: Geodesics on H^2

$SO(3)$. An $SO(1,2)$-transformation that transforms between different points on (2.2) in a Euclidean space would not be a symmetry transformation of the metric, and different points would therefore have different curvature. In Minkowski space-time however the transformation is a symmetry and all points on the surface look the same.

Two things one always should want to know about a space are: 1. Its symmetries, and 2. How its geodesics look like. The sphere has rotational symmetry and its geodesics are its great circles. The symmetries of hyperbolic space I have already described, but what are its geodesics? Remember that there is exactly one geodesic that intersects any given point in a given direction. Consider now the intersection of the hyperboloid with a vertical plane that contains the T-axis. Since the hyperboloid is symmetric with respect to reflections in this plane, geodesics tangential to the plane at some point will be transformed to other geodesics still tangential to the plane at that point by the reflection. Since there can only be one geodesic in a given direction at any point, the reflected geodesic must be the same as the original one and the geodesic must coincide with the vertical plane. Other geodesics can be constructed by boosting or rotating these, which can be realized as the intersection of any time-like plane through the origin with the hyperboloid (see fig 2.1).

2.2 The Poincaré disk

Instead of working with hyperbolic geometry as a surface in a flat embedding space we would like to have a set of intrinsic coordinates. One such coordinate set is obtained by projecting the hyperboloid down on the XY-plane. Choose the point $T=-1$ as the projection point and draw straight lines through the

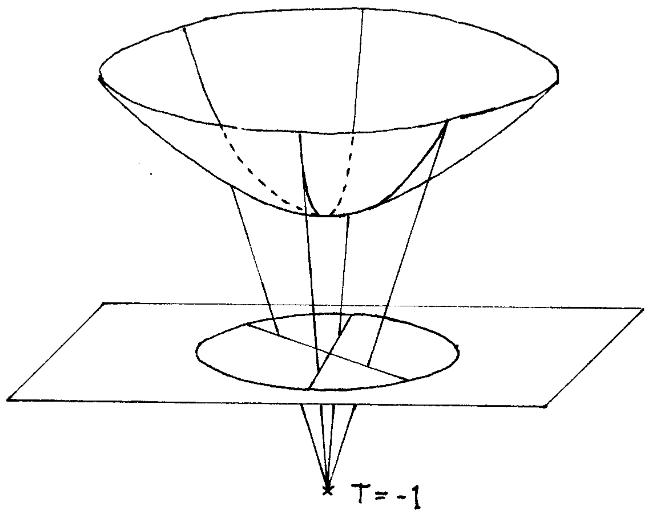

Figure 2.2: Geodesics are projected into straight lines or circle segments.

hyperboloid. The whole space will be projected onto the unit disk, and infinity will be mapped to the unit circle. The metric on the projection plane will be

$$ds^2 = \frac{1}{(1-\rho^2)^2}[d\rho^2 + \rho^2 d\phi^2]\,. \tag{2.3}$$

It is conformal to the flat metric and the mapping is therefore a conformal one. One also sees from the conformal factor that the metric is undefined at $\rho = 1$.

Geodesics on the Poincaré disk are circle segments or straight lines meeting the boundary at right angles (see fig 2.2). In the next section we will see that anti-de Sitter space can be represented as a pile of Poincaré disks, one then has a clear picture of infinity as a time-like cylinder.

2.3 Anti-de Sitter space

3-dimensional anti-de Sitter space can be considered as the surface

$$X^2 + Y^2 - U^2 - V^2 = -1\,, \tag{2.4}$$

embedded in a 4-dimensional pseudo-Riemannian space with metric

$$ds^2 = dX^2 + dY^2 - dU^2 - dV^2. \tag{2.5}$$

We would like to represent anti-de Sitter space in a way that can easily be visualized. A set of intrinsic coordinates that manages this are (t, ρ, ϕ), defined through

$$
\begin{aligned}
X &= \frac{2\rho}{1-\rho^2}\cos\phi\,, \\
Y &= \frac{2\rho}{1-\rho^2}\sin\phi\,, \\
U &= \frac{1+\rho^2}{1-\rho^2}\cos t\,, \\
V &= \frac{1+\rho^2}{1-\rho^2}\sin t\,.
\end{aligned} \tag{2.6}
$$

The whole space is now the interior of the torus $\rho < 1$. Time here is periodic with period 2π. We usually "unwind" the time coordinate and let it range from $-\infty$ to $+\infty$, in which the representation becomes the interior of a cylinder instead. In doing this we are actually dealing with the covering space of anti-de Sitter space, but the distinction is not that important to us.

The line element in these coordinates is

$$
ds^2 = \frac{4}{(1-\rho^2)^2}[d\rho^2 + \rho^2 d\phi^2] - \left(\frac{1+\rho^2}{1-\rho^2}\right)^2 dt^2 = dl^2 - \left(\frac{1+\rho^2}{1-\rho^2}\right)^2 dt^2\,, \tag{2.7}
$$

where dl^2 is the line element on the hyperbolic plane. The metric is undefined at $\rho = 1$. Earlier when we considered the conformal compactification of the 3-sphere, we did so by a conformal transformation of one space to another. Here we have done nothing more than a coordinate change, but the compactification is not complete yet since the metric is undefined at the boundary. Equivalent to a conformal transformation is a conformal rescaling of the metric; we can choose to rescale our metric in such a way that it stays finite at the boundary of the compactified space (denoted $\mathscr{I}$). Introducing a metric on $\mathscr{I}$ in this way with a suitable conformal factor completes the comptacification.

$$
d\hat{s}^2 = \left(\frac{1-\rho^2}{1+\rho^2}\right) ds^2 = \frac{4}{(1+\rho^2)^2}(d\rho^2 + \rho^2\, d\phi^2) - dt^2\,. \tag{2.8}
$$

This gives $\mathscr{I}$ at $\rho = 1$ the flat metric

$$
d\hat{s}^2_{\mathscr{I}} = d\phi^2 - dt^2\,, \tag{2.9}
$$

or in light-cone coordinates

$$
u = t - \phi\,, \quad v = t + \phi\,, \tag{2.10}
$$

$$
d\hat{s}^2 = -du dv. \tag{2.11}
$$

The metric on $\mathscr{I}$ is only a conformal one, that is, it is only defined up to a conformal factor, since we could have chosen any conformal factor that made the metric finite at $\rho = 1$, but this does not matter for the calculation of conformally invariant objects like null geodesics. Conformal metrics will re-appear in another context in the chapter on Einstein-Weyl spaces.

2.4 The isometry group of Anti-de Sitter space

In embedding coordinates, the Killing vectors of anti-de Sitter space are

$$J_{XY} = X\partial_Y - Y\partial_X\,, \qquad J_{XU} = X\partial_U + U\partial_X\,, \tag{2.12}$$

and similarly for the other coordinates.

It will be helpful to us to see that adS_3 can be given a group structure. Manifolds with this property are called group manifolds; the points of the space corresponds to elements in a group. The algebraic structure of the group will help us understand the symmetries of the space. We are a bit lucky to be able to use this property since AdS_3 is the *only* space-time that is also a group manifold.

Consider the group of real 2×2-matrices with determinant one, $SL(2,\mathbb{R})$. A one-to-one correspondence between elements in $SL(2,\mathbb{R})$ and points in AdS_3 can be given by

$$g = \begin{pmatrix} V+X & Y+U \\ Y-U & V-X \end{pmatrix} \quad , \quad X^2 + Y^2 - U^2 - V^2 = -1 \tag{2.13}$$

where (X,Y,U,V) are the coordinates of AdS_3 in the embedding space.

A matrix with determinant one can always be written as the exponent of a traceless matrix m. If we insert a parameter σ in the exponent we obtain a one-parameter subgroup of $SL(2,\mathbb{R})$,

$$g(\sigma) = e^{\sigma m}\,, \tag{2.14}$$

which from another point of view is a curve in AdS_3. At $\tau = 0$, g is the identity element, a very special element in the group but like any other point in the manifold. The tangent vector of the curve at this point is

$$\dot{g}(\sigma)\mid_{\sigma=0} = m\,. \tag{2.15}$$

We see that the tangent space of the manifold around the identity element is the space of traceless matrices. This is the Lie algebra of the group $SL(2,\mathbb{R})$.

Elements of the group can be combined to obtain a new element $g_3 = g_1 g_2$. This makes it possible to view the group not only as the manifold itself but also as a group of transformations on the manifold. A group element g_1 can act on the manifold in two ways; by left translation $g \to g_1 g$, or by right translation $g \to g g_1^{-1}$. We must act with the inverse from the right to ensure that the group structure is preserved, $g g_1^{-1} g_2^{-1} = g(g_2 g_1)^{-1}$. Acting with the subgroup above from the left we obtain a curve through g,

$$g(\sigma) = g_1(\sigma) g = e^{\sigma m} g\,. \tag{2.16}$$

Since g is an arbitrary element of the group this gives us a curve through every point of AdS_3, and because of the group laws the curves can not cross each other, this makes the curves a congruence. The Clifford parallels studied earlier can be constructed like this (S^3 being a group manifold). We can also look at infinitesimal transformations $\delta g = mg$. A basis for the space of traceless matrices is

$$\sigma_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.17}$$

Consider an infinitesimal transformation in the σ_1-direction, $m = \sigma_1$,

$$\begin{pmatrix} \delta(V+X) & \delta(Y+U) \\ \delta(Y-U) & \delta(V-X) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} V+X & Y+U \\ Y-U & V-X \end{pmatrix},. \tag{2.18}$$

The effect of this transformation on the coordinates(X,Y,U,V) can be worked out to be the same as that of the Killing vector $J_{YV} - J_{XU}$,

$$(\delta X, \delta Y, \delta U, \delta V) = (J_{YV} - J_{XU})(X, Y, U, V). \tag{2.19}$$

Doing the same with σ_2 and σ_0 we find that they also correspond to the action of Killing vectors. We then have a basis for the tangent space consisting of three nowhere vanishing Killing vector fields

$$J_0 = J_{XY} - J_{UV}\,, \quad J_1 = J_{YV} - J_{XU}\,, \quad J_2 = -J_{YU} - J_{XV}\,. \tag{2.20}$$

The vector fields are a representation of the Lie algebra $sl(2, \mathbb{R})$. Acting from the right we get another set of fields, commuting with the first since matrix multiplication can be done from the right and left independently.

$$\tilde{J}_0 = -J_{XY} - J_{UV}\,, \quad \tilde{J}_1 = J_{YV} + J_{XU}\,, \quad \tilde{J}_2 = J_{YU} - J_{XV}. \tag{2.21}$$

Together these two sets of vectors generate the symmetry group of AdS_3, $SO(2,2) = SL(2, \mathbb{R}) \otimes SL(2, \mathbb{R})/\mathbb{Z}_2$; each by themselves, they are two sets of bases for the tangent space of $AdS_3 = SL(2, \mathbb{R})$. This dual meaning of the group and its generators is a property shared by all group manifolds equipped with the natural metric. The group can act on itself as a symmetry transformation, and it can do so both from the left and the right independently, giving the group manifold a symmetry group that is the direct product of itself up to a discrete factor.

It is also interesting to see how the Killing fields act on $\mathscr{I}$. Since the metric on $\mathscr{I}$ differs from the metric in the interior by the extra conformal factor they no longer act as true Killing fields but only as conformal ones. That is, the Lie derivative of the metric is proportional to the metric instead of zero. If ξ is a Killing field with respect to the interior metric, g_{ij}, we have on $\mathscr{I}$

$$\pounds_\xi\, \hat{g}_{ij} = \pounds_\xi\, (\Omega^2\, g_{ij}) = g_{ij}\, 2\Omega\, \pounds_\xi \Omega + \Omega^2\, \pounds_\xi g_{ij} = 2\pounds_\xi\, ln\Omega\, \hat{g}_{ij}. \tag{2.22}$$

Listing the conformal Killing fields on $\mathscr{I}$ in light cone coordinates (2.10):

$$\begin{aligned} J_0 &= \partial_v \,, & \tilde{J}_0 &= \partial_u \,, \\ J_1 &= \sin v\, \partial_v \,, & \tilde{J}_1 &= -\sin u\, \partial_u \,, \\ J_2 &= -\cos v\, \partial_v & \tilde{J}_2 &= -\cos u\, \partial_u \,. \end{aligned} \tag{2.23}$$

We can see that all of the generators become light-like on $\mathscr{I}$ and those that were space-like inside anti-de Sitter space have fixed points on the boundary.

In group manifolds it is possible to define *Maurer-Cartan forms*. They are invariant under right (or left) translation and takes values in the Lie algebra. A right invariant Maurer-Cartan form is written like dgg^{-1} and that it is invariant under right translation by a fixed group element g_1 is easily shown:

$$dgg^{-1} \to d(gg_1)(gg_1)^{-1} = dgg_1 g_1^{-1} g^{-1} = dgg^{-1} \,. \tag{2.24}$$

A left invariant Maurer-Cartan form is thus written as $g^{-1}dg$. Since they take values in the Lie algebra they can be expanded in the basis defined above (2.17),

$$dgg^{-1} = \sigma_0\, \Theta_0 + \sigma_1\, \Theta_1 + \sigma_2\, \Theta_2 \,. \tag{2.25}$$

Solving this equation for the Θ_I's we find

$$\begin{aligned} \Theta_0 &= YdX - XdY - VdU + UdV \,, \\ \Theta_1 &= -UdX + VdY + XdU - YdV \,, \\ \Theta_2 &= -VdX - UdY + YdU + XdV \,. \end{aligned} \tag{2.26}$$

These one-forms are dual to the vectors J_I defined above,

$$\Theta_I(J_J) = \delta_{IJ}. \tag{2.27}$$

On a group manifold it is natural to use the Maurer-Cartan forms to define a metric. The natural metric for the classical groups is given by

$$ds^2 = -\frac{1}{2}\mathrm{Tr}(dgg^{-1}dgg^{-1}) = -\Theta_0^2 + \Theta_1^2 + \Theta_2^2. \tag{2.28}$$

Since the trace is invariant under cyclical permutations, the natural metric can also be expanded in the left invariant Maurer-Cartan forms,

$$ds^2 = -\frac{1}{2}\mathrm{Tr}(dgg^{-1}dgg^{-1}) = -\frac{1}{2}\mathrm{Tr}(g^{-1}dgg^{-1}dg) = -\tilde{\Theta}_0^2 + \tilde{\Theta}_1^2 + \tilde{\Theta}_2^2. \tag{2.29}$$

where the $\tilde{\Theta}_I$'s are obtained from the equation

$$g^{-1}dg = \sigma_0\, \tilde{\Theta}_0 + \sigma_1\, \tilde{\Theta}_1 + \sigma_2\, \tilde{\Theta}_2 \,, \tag{2.30}$$

$$\begin{aligned}
\tilde{\Theta}_0 &= -YdX + XdY - VdU + UdV\,, \\
\tilde{\Theta}_1 &= UdX + VdY - XdU - YdV\,, \\
\tilde{\Theta}_2 &= -VdX + UdY - YdU + XdV\,.
\end{aligned} \tag{2.31}$$

and are dual to the $\tilde{J}_I$'s

$$\tilde{\Theta}_I(\tilde{J}_J) = \delta_{IJ}. \tag{2.32}$$

A group manifold G equipped with its natural metric is invariant under both left and right translations and thus has the symmetry group $G \times G$ (up to some discrete factor). The natural metric on $SL(2,\mathbb{R})$ is exactly the metric of AdS_3 (2.5).
With the metric expressed in this form of right or left invariant Maurer-Cartan forms the squashing of AdS_3 to AdS_2 is straight forward, we just have to insert a parameter into the metric in front of one of the space-like generators and turn it down to zero.

$$ds^2 = -\Theta_0^2 + \lambda^2\,\Theta_1^2 + \Theta_2^2\,, \qquad 0 < \lambda < 1. \tag{2.33}$$

We could also, if we wanted to, put λ^2 in front of Θ_0^2 instead.

$$ds^2 = -\lambda^2\,\Theta_0^2 + \Theta_1^2 + \Theta_2^2\,, \qquad 0 < \lambda < 1. \tag{2.34}$$

This squashing of AdS_3 would, when λ is turned to zero, become hyperbolic space, but it is mainly the first squashing we are interested in here. There is still some work to be done, however, before we are ready to describe the squashing. The three vector fields used here to expand the metric do not commute and are therefore not suitable to use for a coordinate system of AdS_3. We would like to have a coordinate system in some way adapted to the squashed space. Working in sausage coordinates or embedding coordinates is possible but cumbersome, and best avoided. Luckily, the group structure of the manifold will help us find the coordinate system we are looking for.

2.5 "Euler coordinates" for AdS_3

To construct a suitable coordinate system, consider the diameter of one of the Poincaré disks in the compact picture of anti-de Sitter space described in section 2.3, say the line $t = 0, \phi = 0, \pi$. This line is both a geodesic and the flow-line of the Killing vector J_1 (and $\tilde{J}_1$) (see fig 2.3).

In the group manifold this line is represented by the set of matrices

$$g = \begin{pmatrix} V+X & Y+U \\ Y-U & V-X \end{pmatrix} = \begin{pmatrix} \frac{2\rho}{1-\rho^2} & \frac{1+\rho^2}{1-\rho^2} \\ -\frac{1+\rho^2}{1-\rho^2} & \frac{-2\rho}{1-\rho^2} \end{pmatrix}, \quad \rho \in (-1,1). \tag{2.35}$$

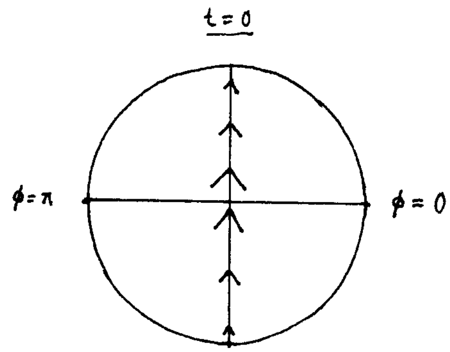

Figure 2.3: A flow line of $\hat{J}_2$ indicates the direction of translation of the line $t = 0$, $\phi = 0, \pi$

Note that ρ ranges from -1 to 1 here in contrast to its normal role as the radial coordinate in the Poincaré disk.

Now, the vector field $\tilde{J}_2$ is orthogonal to the line we just constructed along the whole of its length. We can adapt our coordinate system to this vector field by parameterizing a translation of the line in the direction of $\hat{J}_2$; in the group manifold this translation is simply performed by multiplication of matrices. The translation in the $\tilde{J}_2$-direction of the line corresponds to multiplying the group elements $g(\rho)$ from the right by $e^{-\omega\sigma_2}$, where ω ranges from $-\infty$ to ∞ since $\tilde{J}_2$ generates a non-compact subgroup of the symmetry group of AdS_3.

$$\begin{aligned} g(\rho,\omega) &= \begin{pmatrix} \frac{2\rho}{1-\rho^2} & \frac{1+\rho^2}{1-\rho^2} \\ -\frac{1+\rho^2}{1-\rho^2} & \frac{-2\rho}{1-\rho^2} \end{pmatrix} \begin{pmatrix} e^{-\omega} & 0 \\ 0 & e^{\omega} \end{pmatrix} = \\ & \begin{pmatrix} \frac{2\rho}{1-\rho^2} e^{-\omega} & \frac{1+\rho^2}{1-\rho^2} e^{\omega} \\ -\frac{1+\rho^2}{1-\rho^2} e^{-\omega} & \frac{-2\rho}{1-\rho^2} e^{\omega} \end{pmatrix} . \end{aligned} \tag{2.36}$$

Keeping the cylinder picture of infinity in mind, and the action of $\tilde{J}_2$ on it (2.23), we can see that the endpoints of the line $g(\rho)$ each will be translated in a light-like direction around the cylinder $\pm\frac{\pi}{2}$ in the u-direction, and that each end is translated in the opposite direction of the other. The ends end up on the same line of fixed points of the vector field $\tilde{J}_2$ on the boundary, $u = \pm\frac{\pi}{2}$ respectively, for translation in positive or negative $\tilde{J}_2$-direction. In fact, each point of the line $g(\rho)$ will end up on the same line of fixed points for an infinite translation in the $\tilde{J}_2$-direction. The line is light-like which makes the section $\mathrm{g}(\rho,\omega)$ a space-like section of AdS_3 with a light-like boundary, where the rest of the boundary is made up of the flow-lines of $\tilde{J}_2$ containing the endpoints of $g(\rho)$. The boundary has four kinks at $(u, v) = (-\frac{\pi}{2}, 0), (-\frac{\pi}{2}, \pi), (\frac{\pi}{2}, 0), (\frac{\pi}{2}, \pi)$ (fig 2.4).

All we have to do now to complete our coordinate basis is to find a third everywhere non-vanishing vector field that commutes with the previous two. Actually, we already have such a field. The time-like Killing vector from the set that corresponds to left translation, J_0, commutes with $\tilde{J}_2$, and is actually

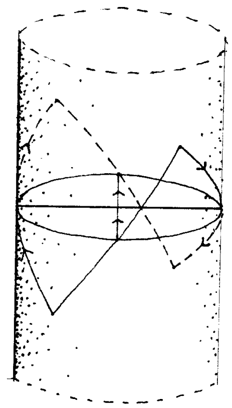

Figure 2.4: The bundary of the coordinate surface parameterized by ρ and ω. Arrows indicate the flow of $\hat{J}_2$

orthogonal to the line $g(\rho)$ which means that it is orthogonal to all lines $g_\omega(\rho)$ that is a result of the translation by $\tilde{J}_2$ since this translation is a conformal transformation. It is not orthogonal to $\tilde{J}_2$, hence our coordinate system will not be an orthogonal one. Translating the surface $g(\rho, \omega)$ along J_0 gives us

$$g(\rho,\omega,\tau) = \begin{pmatrix} \cos\tau & -\sin\tau \\ \sin\tau & \cos\tau \end{pmatrix} \begin{pmatrix} \frac{2\rho}{1-\rho^2} e^{-\omega} & \frac{1+\rho^2}{1-\rho^2} e^{\omega} \\ -\frac{1+\rho^2}{1-\rho^2} e^{-\omega} & \frac{-2\rho}{1-\rho^2} e^{\omega} \end{pmatrix} =$$
$$\frac{1}{1-\rho^2} \begin{pmatrix} (2\rho\, \cos\tau + (1+\rho^2)\sin\tau) e^{-\omega} & \big((1+\rho^2)\cos\tau + 2\rho\, \sin\tau\big) e^{\omega} \\ (2\rho\, \sin\tau - (1+\rho^2)\cos\tau) e^{-\omega} & \big((1+\rho^2)\sin\tau - 2\rho\, \cos\tau\big) e^{\omega} \end{pmatrix}. \tag{2.37}$$

From this we can read off the relation between the intrinsic coordinates (ρ, ω, τ) and the embedding coordinates (X,Y,U,V) by comparing with equation (2.35). If we at the same time define σ through $\rho = \tanh(\frac{\sigma}{4})$ and rescale the other coordinates a bit we arrive at a nice coordinate system of AdS_3.

$$\begin{aligned} X &= +\sinh\left(\frac{\sigma}{2}\right) \cosh\left(\frac{\omega}{2}\right) \cos\left(\frac{\tau}{2}\right) - \cosh\left(\frac{\sigma}{2}\right) \sinh\left(\frac{\omega}{2}\right) \sin\left(\frac{\tau}{2}\right), \\ Y &= +\sinh\left(\frac{\sigma}{2}\right) \cosh\left(\frac{\omega}{2}\right) \sin\left(\frac{\tau}{2}\right) + \cosh\left(\frac{\sigma}{2}\right) \sinh\left(\frac{\omega}{2}\right) \cos\left(\frac{\tau}{2}\right), \\ U &= +\cosh\left(\frac{\sigma}{2}\right) \cosh\left(\frac{\omega}{2}\right) \cos\left(\frac{\tau}{2}\right) + \sinh\left(\frac{\sigma}{2}\right) \sinh\left(\frac{\omega}{2}\right) \sin\left(\frac{\tau}{2}\right), \\ V &= +\cosh\left(\frac{\sigma}{2}\right) \cosh\left(\frac{\omega}{2}\right) \sin\left(\frac{\tau}{2}\right) - \sinh\left(\frac{\sigma}{2}\right) \sinh\left(\frac{\omega}{2}\right) \cos\left(\frac{\tau}{2}\right). \end{aligned} \tag{2.38}$$

The three vector fields $\partial_\sigma, \partial_\omega$ and ∂_τ are all commuting and linearly independent *inside* AdS_3 and are therefore suitable for use as a coordinate basis, but on $\mathscr{I}$

∂_σ and ∂_τ are both proportional to ∂_v (remember that the infinitely translated line $g(\rho)$ coincides with the line of fixed points of $\tilde{J}_2$) and hence they are no longer linearly independent. This will be important when we want to look at $\mathscr{I}$ of the squashed space.

We finish this section with the Killing fields and the metric in our new coordinates.

$$
\begin{array}{rcl}
 & & \textit{Killing vectors} \\
J_0 & = & 2\partial_\tau \, , \\
J_1 & = & 2\sin\tau\ \tanh\sigma\,\partial_\tau - 2\ \cos\tau\,\partial_\sigma + 2\sin\tau\,\mathrm{sech}\,\sigma\,\partial_\omega \, , \\
J_2 & = & -2\cos\tau\ \tanh\sigma\,\partial_\tau - 2\ \sin\tau\,\partial_\sigma - 2\cos\tau\,\mathrm{sech}\,\sigma\,\partial_\omega \, , \\
\tilde{J}_0 & = & 2\mathrm{sech}\,\sigma\ \cosh\omega\,\partial_\tau + 2\sinh\omega\,\partial_\sigma - 2\tanh\sigma\ \cosh\omega\,\partial_\omega \, , \\
\tilde{J}_1 & = & 2\mathrm{sech}\,\sigma\ \sinh\omega\,\partial_\tau + 2\cosh\omega\,\partial_\sigma - 2\tanh\sigma\ \sinh\omega\,\partial_\omega \, , \\
\tilde{J}_2 & = & 2\partial_\omega \, .
\end{array}
\tag{2.39}
$$

$$ds^2 = \frac{1}{4}(-d\tau^2 + d\sigma^2 + d\omega^2 + 2\ \sinh\sigma\, d\tau d\omega). \tag{2.40}$$

We can compare this with the metric of the 3-sphere in Euler angles [1].

$$ds^2_{S^3} = \frac{1}{4}(d\tau^2 + d\theta^2 + d\phi^2 + 2\ \cos\theta d\tau d\phi) \tag{2.41}$$

where

$$0 \leq \tau < 4\pi \qquad 0 \leq \phi < \pi \qquad 0 < \theta < \pi. \tag{2.42}$$

With this we have a description of anti-de Sitter space adapted to the symmetries we will use when we are squashing the space, similar to the fibre bundle description of the 3-sphere.

Chapter 3

Squashing Anti-de Sitter space

Since anti-de Sitter space is a space-time and has both time-like and space-like directions, it is possible to obtain quite different results by choosing to squash in either the time-like or a space-like direction. The two different ways of squashing can, as we saw in the previous chapter, be realized by either inserting a parameter in the metric in front of a time-like, or a space-like one-form and generator of the group AdS_3. It is mainly the squashing in the space-like direction that will be described here, but I will spend a few words on the other case too. A description of the time-like squashing can also be found in [7].

In the previous chapter we made some preparations by giving a fibre bundle description of the space where the fibre coordinate σ runs along the space-like fibre we intend to squash. The coordinate system is actually adapted to both cases at the same time, the coordinate τ runs along a time-like fibre related to the sausage-coordinates as $\tau = t + \phi = v$ (2.23). ∂_τ generates a subgroup isomorphic to $SO(2)$ while ∂_σ is the generator of the non-compact subgroup $SO(1,1)$. The resulting quotient spaces if we were to factor the groups out are different, as we will see.

In the time-like case we can rewrite the metric (2.40) in the form

$$ds^2 = \frac{1}{4}(-(d\tau - \sinh\sigma d\omega)^2 + d\sigma^2 + \cosh^2\sigma d\omega^2). \tag{3.1}$$

$(d\tau - \sinh\sigma d\omega)$ is the the one-form Θ_0 (2.31) expressed in our new coordinates. The metric of (temporally-) squashed anti-de Sitter space is obtained by inserting a parameter in front of Θ_0.

$$ds^2_\lambda = \frac{1}{4}(-\lambda^2\,(d\tau - \sinh\sigma d\omega)^2 + d\sigma^2 + \cosh^2\sigma d\omega^2). \tag{3.2}$$

When $\lambda = 0$ we obtain a two-dimensional space parametrized by the coordinates (σ, ω), with metric

$$ds^2_{\lambda=0} = d\sigma^2 + \cosh^2\sigma d\omega^2. \tag{3.3}$$

This metric describes a space of constant negative curvature - the hyperbolic plane. Thus we can write $H_2 = AdS_3/SO(2)$.

Let us now look at the other case, the squashing along the space-like fibre generated by $\tilde{J}_2$. First we rewrite the metric (2.40) in the form

$$ds^2 = \frac{1}{4}(-\cosh^2\sigma d\tau^2 + (d\omega + \sinh\sigma d\tau)^2 + d\sigma^2). \tag{3.4}$$

$(d\phi + \sinh^2\omega dt)$ is the one-form $\tilde{\Theta}_2$, dual to $\tilde{J}_2$. The metric of (spatially-)squashed anti-de Sitter space is obtained by inserting λ^2 in front of $\tilde{\Theta}_2$.

$$ds_\lambda^2 = \frac{1}{4}(-\cosh^2\sigma d\tau^2 + \lambda^2(d\omega + \sinh\sigma d\tau)^2 + d\sigma^2)\,. \tag{3.5}$$

By letting λ range between 0 and 1 we have a family of metrics that interpolate between 3-dimensional and 2-dimensional anti-de Sitter space. To convince ourselves that we do actually arrive at 2-dimensional anti-de Sitter space let us look at the metric when $\lambda = 0$:

$$ds_{\lambda=0}^2 = \frac{1}{4}(-\cosh^2\sigma d\tau^2 + d\sigma^2)\,. \tag{3.6}$$

This is one fourth of the metric of 2D-anti-de Sitter space in a commonly used set of coordinates that can simply be related to a one sheeted hyperboloid defined in 3D-Minkowski space

$$x^2 - u^2 - v^2 = -1\,, \qquad ds^2 = -(du)^2 - (dv)^2 + (dx)^2 \tag{3.7}$$

if we parametrize it as

$$\begin{aligned} u &= \cosh\sigma\,\cos\tau\,, \\ v &= \cosh\sigma\,\sin\tau\,, \\ x &= \sinh\sigma\,. \end{aligned} \tag{3.8}$$

The coordinates are static and global.

3.1 Symmetries and Killing Horizons

3-dimensional anti-de Sitter space is a well known space-time of constant negative curvature and a solution to Einstein's equations in three dimensions, but what are these geometries that we obtain when we squash it? One obvious consequence of the squashing is the reduction of symmetry. 3D-anti-de Sitter space was a maximally symmetric space while the squashed space has a preferred direction along the fibre we choose for the squashing. Of the six Killing vectors of anti-de Sitter space, (2.20), (2.21), only four still describe symmetries of the space - those that commuted with the vector of the squashing direction. The remaining Killing vectors are the three vectors describing left translation on the group manifold, and the squashing vector itself.

$$\{\xi_{Killing}\} = \{J_0,\, J_1,\, J_2,\, \tilde{J}_2\}\,. \tag{3.9}$$

Thus the symmetry group of squashed anti-de Sitter space is $SL(2,\mathbb{R})\otimes SO(1,1)$, since $\tilde{J}_2$ generates the group $SO(1,1)$, the set of vectors corresponding to left translation generates $SL(2,\mathbb{R})$ and $\tilde{J}_2$ commutes with the other three. The subgroup $SL(2,\mathbb{R})$ of the total symmetry group translates between all points in the manifold, thus the group is homogeneous, but not isotropic.

Killing horizons are null-surfaces whose generating null-vector coincides with a Killing vector field [3]. The physical significance of such a surface is that on it a particle can travel at the speed of light and still seem to be standing still; it is traveling at the speed of light if its trajectory follows the null generators, but standing still in the sense that no change in its surroundings can be detected as this trajectory at the same time is the flow-line of a Killing field (and fixed in relation to the surface).

In anti-de Sitter space, the Killing vector $\xi = J_{XU}$ generates a Killing horizon. The surface where it becomes null,

$$\xi^\alpha \xi_\alpha = \|J_{XU}\|^2 = U^2 - X^2 = (U+X)(U-X) = 0\,, \tag{3.10}$$

is a null surface consisting of the null planes $X = U$ and $X = -U$. Naturally also J_{XV}, J_{YU} and J_{YV} generate identical Killing Horizons of two intersecting null planes. If one instead considers the Killing vector that is a combination of a rotation and a boost, $\xi = (J_{XU} + J_{XY})$, the surfaces where it becomes null,

$$\xi^\alpha \xi_\alpha = \|J_{XU} + J_{XY}\|^2 = (U+Y)^2 = 0\,, \tag{3.11}$$

consists of one single sheet, and the normal to the surface is the zero vector,

$$\nabla_\alpha (U+Y)^2 = 0\,. \tag{3.12}$$

This is called a degenerated Killing horizon.

Much of the symmetries remain in squashed anti-de Sitter space and with them the Killing horizons they generate, but many do disappear. The generating vectors of the two-sheeted Killing horizons are constructed by adding the basis vectors of the symmetry group (2.20), (2.21) together, one from each mutually commuting set. Since two of the basis vectors of the set corresponding to right translation no longer are Killing vectors ($\tilde{J}_0$, $\tilde{J}_1$), only the combinations $J_{YU} = \frac{\tilde{J}_2 - J_2}{2}$ and $J_{XV} = -\frac{\tilde{J}_2 - J_2}{2}$ still generate Killing horizons. Of the one-sheeted Killing horizons generated by the rotation-boost vectors none remain, this because the rotation vectors J_{XY} and J_{UV} no longer act as isometries. One would expect that with the reduction of symmetry there would be a reduction of Killing horizons too, but curiously, a new horizon emerges when we start to squash the space. The vector $J_0 + J_1$ is light-like everywhere in anti-de Sitter space,

$$
\begin{aligned}
J_0 + J_1 &= J_{XY} - J_{UV} + J_{UV} - J_{XU} = \\
&= (Y-U)\partial_X + (V-X)\partial_Y + (V-X)\partial_U + (Y-U)\partial_V \\
\|J_0 + J_1\|^2 &= (Y-U)^2 + (V-X)^2 - (V-X)^2 - (Y-U)^2 = 0!
\end{aligned}
\tag{3.13}
$$

Rewriting the vector in intrinsic coordinates and calculating the norm in the squashed space we find

$$
\begin{aligned}
J_0 + J_1 &= (2 + 2\sin\tau\ \tanh\sigma)\partial_\tau - 2\cos\tau\,\partial_\sigma + 2\sin\tau\,\mathrm{sech}\,\sigma\,\partial_\omega \\
\|J_0 + J_1\|^2 &= (\lambda^2 - 1)(\sin\tau\ \cosh\sigma + \sinh\sigma)^2
\end{aligned}
\tag{3.14}
$$

It is only light-like on the surface $(\sin\tau\ \cosh\sigma + \sinh\sigma)^2 = 0$ which has a null normal for the same reason as the earlier single-sheeted Killing horizons did (a surface $f^2(\mathbf{r}) = 0$ has normal vector $\nabla f^2(\mathbf{r}) = f(\mathbf{r})\nabla f(\mathbf{r}) = \mathbf{0}$), hence it is a degenerate Killing horizon. It would be interesting to do a more systematic investigation of the Killing horizons of squashed anti-de Sitter space but due to restrictions of time none will be done here.

3.2 Conformal infinity of squashed anti-de Sitter space

What about the boundary then? The space we started with had a conformal boundary in the form of a cylinder. The space we end up with when the squashing is complete has a boundary consisting of two time-like lines. Does the squashing just gradually deform the cylinder until it is flat, leaving the boundary more or less intact, or does something else happen? Since $\tilde{J}_2$'s action on $\mathscr{I}$ is known (2.23), it is at first sight natural to assume that the squashing of $\mathscr{I}$ can be done independently from the squashing of the interior, just tweak the metric on $\mathscr{I}$ in the same way as we did with the interior metric. But some further thought should make us a bit cautious about this conclusion. It is in no way obvious that there should be a continuous transition between the old $\mathscr{I}$ and the new one as it is constructed through a limiting procedure. The conformal factor used before may no longer produce a finite metric at infinity, instead some other conformal rescaling could be needed to bring the metric under control, with the effect that the structure of $\mathscr{I}$ dramatically changes. To see what actually happens we should reconstruct $\mathscr{I}$ from the squashed metric, we may or may not end up at a cylindrical $\mathscr{I}$ near $\lambda = 1$, but it is impossible to know which beforehand. Since squashed AdS_3 is described throughout this thesis with the coordinates adapted to its symmetries we should first try to conformally compactify the original 3-dimensional anti-de Sitter space in these coordinates so that we later can compare it with the compactified squashed space.

Thus, looking at the metric (2.40), we can see that it blows up at large positive and negative σ. We can make it finite by a rescaling with the factor $2/\sinh\sigma$ and then let σ go to infinity.

$$d\hat{s}^2 = \frac{2}{\sinh\sigma}ds^2 = \frac{1}{2\sinh\sigma}(-d\tau^2 + d\sigma^2 + d\omega^2 + 2\sinh\sigma d\tau d\omega)\,, \tag{3.15}$$

$$\lim_{\sigma\to\pm\infty} d\hat{s}^2 \quad = \quad d\tau d\omega\,. \tag{3.16}$$

The resulting $\mathscr{I}$ is flat and τ and ω become light-like coordinates as we would expect them to be from (2.23). It may seem odd to use only one of the two space-like coordinates to reach infinity, like compactifying in the x-direction of a Euclidean space but not the y-direction, but it is actually not. First of all, we do actually *know* what infinity looks like in this case, we have done nothing to the metric except a coordinate change and $\mathscr{I}$ must still be a cylinder. Moreover, we also know how the coordinate vectors behave on $\mathscr{I}$ in the cylinder picture. When we travel to $\pm\sigma$-infinity we end up on one of two light-like lines each reaching π u-coordinates around the cylinder. We still have the freedom to move along these lines in the ω-direction ($\propto u$-direction on $\mathscr{I}$) or the opposite light-like $\tau(v)$-direction. This freedom of movement will allow us to reach every point on the cylinder. It will also allow us to travel between the apparently disjoint patches $\mathscr{I}_\pm$(corresponding to $\pm\sigma$ infinity) which actually lie only π v-coordinates apart. The extension of the lines in the τ-direction turn them into two light-like bands that wrap around the cylinder, joined together at the lines of fixed points of $\tilde{J}_2$, i.e. at $\pm\omega$-infinity, or in the cylinder-light cone coordinates, at $u = \pm\pi/2$.

If we now look at the squashed space instead

$$\begin{aligned} ds^2_\lambda &= \frac{1}{4}(-\cosh^2\sigma d\tau^2 + \lambda^2(d\omega + \sinh\sigma d\tau)^2 + d\sigma^2) \\ &= \frac{1}{4}[-(\cosh^2\sigma - \lambda^2\sinh^2\sigma)d\tau^2 + d\sigma^2 + \lambda^2 d\omega^2 + 2\lambda^2\sinh\sigma d\tau d\omega]\,. \end{aligned} \tag{3.17}$$

we see that a factor of $2/\sinh\sigma$ will not be enough to make the metric finite when we let σ go to infinity, instead we will need something of the order $e^{-2\sigma}$. Here we will use $1/cosh^2\sigma$ to try to construct a $\mathscr{I}$.

$$\begin{aligned} d\hat{s}^2 &= \frac{1}{\cosh^2\sigma}ds^2 = \\ &= \frac{1}{4}[-(1-\lambda^2\tanh^2\sigma)d\tau^2 + \frac{d\sigma^2}{\cosh^2\sigma} + \frac{\lambda^2 d\omega^2}{\cosh^2\sigma} + \frac{\lambda^2\tanh\sigma d\tau d\omega}{\cosh\sigma}]\,, \end{aligned} \tag{3.18}$$

$$\lim_{\sigma\to\pm\infty} d\hat{s}^2 \quad = \quad \frac{1}{4}(-(1-\lambda^2)d\tau^2 + 0\cdot d\tau d\omega + 0^2\cdot d\omega^2). \tag{3.19}$$

When σ goes to infinity the conformal metric becomes degenerate. This indicates that the boundary of our squashed space is lightlike, in contrast to the time-like cylinder of anti-de Sitter space. A finite metric is not enough to construct $\mathscr{I}$ however, a successfully compactified space should be extendable beyond $\mathscr{I}$ in such a way that the original space is only a part of a larger space. For this to work we must make sure that the curvature on $\mathscr{I}$ is finite everywhere.

Calculating the Ricci scalar of the conformally related metric $\hat{g}$ we find that it is exponential in σ, $R = 8 + 2\lambda^2 \cosh^2 \sigma$, and hence not finite at $\mathscr{I}$! It seems like squashed anti-de Sitter space can not be conformally compactified. To be absolutely sure one should try to compactify the space with a general conformal factor and see if there is any choice for which the curvature on $\mathscr{I}$ stays finite. This is in stark contrast to space-times that are solutions to Einstein's equation. For such space-times Roger Penrose once showed that the structure of $\mathscr{I}$ essentially depends on the sign of the cosmological constant. A space-time with positive cosmological constant has a $\mathscr{I}$ that is space-like, a negative cosmological constant gives a time-like $\mathscr{I}$ and a space-time without a cosmological constant has a $\mathscr{I}$ that is null, like Minkowski space does. Our space-time is of course a solution to the Einstein equation, but with an unphysical energy-momentum tensor that do not fall of at infinity, which is assumed in Penrose's argument. Leaving the constraints the Einstein equation places on the space-time behind us, we find that not only should we be careful to make assumptions of the structure of $\mathscr{I}$ based on arguments that are no longer valid, but also that the very existence of a $\mathscr{I}$ becomes uncertain.

Chapter 4

Einstein-Weyl spaces

Squashed anti-de Sitter space is not a solution to Einstein's vacuum equations, but there is a more general set of equations, called the Einstein-Weyl equations, to which our space might be a solution. The Einstein-Weyl equations are a conformally invariant generalization of the Einstein equations. In three dimensions the Einstein equations restrict the space to having constant curvature, while the Einstein-Weyl equations allow the spaces greater freedom. While the local structure of three-dimensional Einstein spaces are characterized by the single constant number, the Ricci scalar, it has been shown by Cartan [7] that the Einstein-Weyl spaces need four arbitrary functions of two variables to be fully specified, indicating the larger class of geometries they constitute. This chapter gives a very brief introduction to Einstein-Weyl spaces, for more substantial treatments of the subject the reader is referred to [7], [10] and [6].

4.1 Definition of Weyl spaces and the Einstein-Weyl equation

A Weyl space can be defined as a smooth manifold $\mathcal{W}$ together with

1. a conformal metric
2. a torsion-free covariant derivative (called the Weyl connection)

where the metric is preserved by the connection.

A conformal metric is actually a class of metrics related by a conformal factor. The condition that the metric is preserved by the connection means that if g_{ij} is a choice of metric from the given class and D_i is the connection, then

$$D_i\, g_{jk} = \omega_i\, g_{jk}\,, \tag{4.1}$$

where ω_i is a one-form. The use of another metric from the conformal class is equivalent to a rescaling of the first with an everywhere non-vanishing conformal factor Ω^2.

$$g_{ij} \to \hat{g} = \Omega^2 g_{ij}\,. \tag{4.2}$$

Since the metric must be preserved by the connection, the one-form is required to transform under the rescaling like

$$\omega_i \to \hat{\omega}_i = \omega_i + 2\nabla_i \, ln \, \Omega \, . \tag{4.3}$$

∇_i is the metric Levi-Civita connection of the chosen metric g_{ij}. The Weyl and Levi-Civita connections are related by

$$D_i \, V^j = \nabla_i \, V^j + \gamma^j{}_{ik} \, V^k \, , \tag{4.4}$$

where $\gamma^j{}_{ik}$ can be expressed in terms of ω_i,

$$\gamma^j{}_{ik} = -\frac{1}{2}(\delta^j_i \omega_k + \delta^j_k \omega_i - g_{ik} g^{jm} \omega_m). \tag{4.5}$$

From the connection we can construct many of the objects normally associated with curved spaces; the curvature tensor,

$$(D_i D_j - D_j D_i) V^k = W^k{}_{mij} \, V^m \, , \tag{4.6}$$

the Ricci tensor,

$$W_{ij} = W^m{}_{imj} \, , \tag{4.7}$$

the Ricci scalar,

$$W = g^{ij} \, W_{ij} \, , \tag{4.8}$$

where we use any of the metrics in the conformal class to raise and lower indices. This means that while the equations themselves are invariant under (4.2) the objects constructed from D_i and g_{ij} are not necessarily so. For example $W^i{}_{jkl}$ and W_{ij} are invariant under (4.2) while W transforms like $W \to \Omega^{-2} W$. The Weyl Ricci tensor is not symmetric, and in dividing it into symmetric and anti-symmetric parts we find (in three dimensions)

$$W_{[ij]} = -\frac{3}{2} \nabla_{[i} \omega_{j]} \tag{4.9}$$

and

$$W_{(ij)} = R_{ij} + \frac{1}{2} \nabla_{(i} \omega_{j)} + \frac{1}{4} \omega_i \omega_j + g_{ij} (\frac{1}{2} \nabla_k \omega^k - \frac{1}{4} \omega_k \omega^k) \, , \tag{4.10}$$

where the symmetric part of W_{ij} has been expressed in terms of the Ricci tensor of the Levi-Civita connection.

In an Einstein-Weyl space the symmetric part of the Weyl Ricci tensor is proportional to the metric

$$W_{(ij)} = \frac{1}{3} W g_{ij} \,. \tag{4.11}$$

This is the Einstein-Weyl equation, which is a natural conformally invariant generalization of the Einstein equation. Rewritten in terms of the Levi-Civita connection and ω_i it looks like

$$R_{ij} + \frac{1}{2}\nabla_{(i}\omega_{j)} + \frac{1}{4}\omega_i\omega_j = \Lambda g_{ij} \tag{4.12}$$

where Λ is a scalar function.

4.2 An interesting property

It can be shown that every Einstein-Weyl space, or its complexification, contains a two-parameter family of totally geodesic null hypersurfaces [7]. In Minkowski space these null hypersurfaces are the null-planes. The null planes of Minkowski space can also be considered as light cones with vertex on $\mathscr{I}$, thus the two-parameter family of totally geodesic null hypersurfaces can in this case be put in direct correspondence to $\mathscr{I}$. This is also true for anti-de Sitter space. It is not true for generic space-times with matter where light cones can behave in all manner of complicated ways and do not form hypersurfaces at all; and it is not clear to me what the requirements on a space-time are to be able to exhibit this correspondence, but it at least hints at a possibly interesting connection.

The Berger sphere mentioned earlier is an example of an Einstein-Weyl space [10]. One can specify an Einstein-Weyl space by giving a metric $\mathbf{g}$ and a one-form ω from the conformal class. The metric and one-form are here expressed in the standard basis one-forms on S^3.

$$\begin{aligned} g &= \sigma_1^2 + \sigma_2^2 + \lambda^2\sigma_3^2 \,, \\ \omega &= \pm 4\lambda\sqrt{1-\lambda^2}\,\sigma_3 \,, \end{aligned} \tag{4.13}$$

or if expressed in Euler coordinates,

$$\begin{aligned} ds^2 &= \frac{1}{4}(\lambda^2\,(d\tau + \cos\theta\, d\phi)^2 + d\theta^2 + \sin^2\,\theta d\phi^2) \,, \\ \omega &= \pm 4\lambda\sqrt{1-\lambda^2}\,(d\tau + \cos\theta\, d\phi) \,. \end{aligned} \tag{4.14}$$

Another Einstein-Weyl space is the temporally-squashed anti-de Sitter space [7].

$$\begin{aligned} g &= -\lambda^2\Theta_0^2 + \Theta_1^2 + \Theta_2^2 \,, \\ \omega &= \pm 4\lambda\sqrt{1-\lambda^2}\,\Theta_0 \,, \end{aligned} \tag{4.15}$$

in our coordinate system,

$$
\begin{aligned}
ds^2_\lambda &= \frac{1}{4}(-\lambda^2\,(d\tau - \sinh\sigma d\omega)^2 + d\sigma^2 + \cosh^2\,\sigma d\omega^2)\,, \\
\omega &= \pm\,4\lambda\,\sqrt{1-\lambda^2}\,(d\tau - \sinh\sigma d\omega)\,. \qquad (4.16)
\end{aligned}
$$

The one-form ω (4.1) is proportional to the basis one-form along which the space is squashed in both cases. This should lead us to believe that squashed anti de-Sitter space is an Einstein-Weyl space too, since the construction is identical. If it is, then we know it has a 2-parameter family of totally geodesic null hypersurfaces, and we could investigate if there is any connection between this family and the existence of $\mathscr{I}$ on squashed AdS_3.

As it turns out though, squashed anti-de Sitter space is an Einstein-Weyl space only for values of λ greater than one. With the metric expressed in the left-invariant one-forms $\tilde{\Theta}_i$ and using the ansatz $\omega = f(\lambda)\,\tilde{\Theta}_2$ in the Einstein-Weyl equation (4.11) we find the solutions

$$
\begin{aligned}
g &= -\tilde{\Theta}_0^2 + \tilde{\Theta}_1^2 + \lambda^2\,\tilde{\Theta}_2^2\,, \\
\omega &= \pm\,4\lambda\,\sqrt{\lambda^2-1}\,\Theta_2 \qquad (4.17)
\end{aligned}
$$

or in our coordinates,

$$
\begin{aligned}
ds^2_\lambda &= \frac{1}{4}(-\cosh^2\,\sigma d\tau^2 + \lambda^2(d\omega + \sinh\sigma d\tau)^2 + d\sigma^2)\,, \\
\omega &= \pm\,4\lambda\,\sqrt{\lambda^2-1}\,(d\omega + \sinh\sigma d\tau)\,. \qquad (4.18)
\end{aligned}
$$

The sign in the square root is reversed which means that the one-form ω is defined only for values of $\lambda^2 > 1$. This is a bit of a disappointment since at least at first sight values of λ between zero and one seem seem more interesting.

Finally we note that Penrose's argument for the relationship between the sign of the cosmological constant and the signature of the metric on $\mathscr{I}$ can not be repeated with the Einstein-Weyl equation in place of the Einstein equation, this because of the conformal invariance of the Einstein-Weyl equation.

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