Mutually Unbiased Bases for Continuous Variables

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Abstract

The generalization of Mutually Unbiased (MU) bases to continuous variables is introduced and results by Weigert and Wilkinson [5] are examined and extended. It is suggested that the state overlap between the generalized eigenstates of position and momentum has a direct physical meaning, by arguing that these states can be approximated by squeezed states (for which the state overlap is a physically meaningful quantity) and showing that the overlap between such states behaves characteristically like that of the generalized states under transformations in phase space. Thus, regarding generalized position/momentum eigenstates as a limiting case of squeezed states gives physical meaning to their overlap, which in turn implies that the condition that the bases must be configured as the triples introduced in [5] to be MU is physically meaningful. It is also shown that the two different triples introduced by Weigert and Wilkinson, the symmetric and asymmetric triples, are related by a transformation analogous to a Lorentz-boost in phase space; demonstrating that these two triples are not unique.

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1 Introduction

In Quantum Mechanics, a particle, such as a photon, exists in a state which may be a superposition of other states. This superposition can be quantified by selecting a basis, a set of states which the total state can be expressed as a linear combination of. For example, the polarization state of a photon can be described by selecting two vectors in a plane; it's then possible to specify an arbitrary polarization vector as a linear combination of these. A common choice of polarization basis is to select the orthogonal states $|H\rangle$ and $|V\rangle$, corresponding to the photon being polarized horizontally or vertically with respect to some reference axis. An arbitrary state is then given by

$$|\Psi\rangle = \alpha\,|H\rangle + \beta\,|V\rangle \quad \text{where} \quad \alpha^2 + \beta^2 = 1 \tag{1.1}$$

If a photon in the state $|\Psi\rangle$ arrives at some optical component that diverts horizontally and vertically polarized photons in different directions, and photon detectors are placed after this component such that one can tell which way the photon went, then the state has been measured in the given basis. The probability that the photon will be detected as being in the state $|H\rangle$ or $|V\rangle$ is then given by $|\langle H|\Psi\rangle|^2 = \alpha^2$ and $|\langle V|\Psi\rangle|^2 = \beta^2$, respectively.

This choice of basis is of course arbitrary. One could aswell construct a measuring device that differentiates between the following states:

$$|D\rangle = \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle) \quad \text{and} \quad |\bar{D}\rangle = \frac{1}{\sqrt{2}}(|H\rangle - |V\rangle) \tag{1.2}$$

Corresponding to diagonal and anti-diagonal polarization with respect to the H-V basis. This device would then perform a measurement in the D-$\bar{D}$ basis.

If a photon is prepared in such a way that it is *known* to be in the diagonally polarized state

$$|\Psi\rangle = |D\rangle = \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle) \tag{1.3}$$

then a measurement in the D-$\bar{D}$ basis will always find that the photon is diagonally polarized. If the same measurement is instead performed in the H-V basis, then it will be found to be in one of the states $|H\rangle$ or $|V\rangle$, with probabilities $|\langle H|\Psi\rangle|^2 = 1/2$ and $|\langle V|\Psi\rangle|^2 = 1/2$ respectively.

In other words, a state that is uniquely determined in the D-$\bar{D}$ basis is completely *undetermined* in the H-V basis. Bases having this property are called *Mutually Unbiased* (MU) with respect to each other, and this is just one example of such bases. Note that this is not a property of a basis on it's own, but a shared property between two or more bases. It is an extremely useful property which can be exploited in many applications. A simple example would be using these bases to set up measurements that are completely random, by preparing a state in one basis and measuring it in a basis that is MU with respect to the former. This can be useful for constructing random number generators, for example.

Perhaps more significantly, these bases are important in quantum communication and quantum encryption. Quantum key distribution, which is a method used to securely share a classical encryption key between two parties over a quantum channel (such as via exchance of photons), utilizes MU bases to achieve maximum security. This allows the sharing parties to detect if the key has been intercepted by an eavesdropper, in which case the key is not used to encrypt the real message. Many quantum key distribution protocols utilize two MU bases of a two-state system [1], but these methods can be generalized to higher-dimensional systems using more than two MU bases, which gives better security. [2]

The example bases demonstrated above belong to the two-dimensional Hilbert space of photon polarization states. Mutually unbiased bases can be defined more formally, for Hilbert spaces of arbitrary dimensions, in the following way: In a finite-dimensional Hilbert space $\mathbb{C}^d$ of dimension d, an orthonormal basis $\{|e_i\rangle\}_{i=1}^{d}$ is said to be mutually unbiased (MU) with respect to another basis $\{|f_i\rangle\}_{i=1}^{d}$ if the basis vectors satisfy the condition [3]

$$\left|\langle e_i | f_j \rangle\right|^2 = \frac{1}{d} \qquad \forall\, i, j \in [1, d] \tag{1.4}$$

If the state of a system is measured in such a basis, any subsequent measurement performed in a basis that is mutually unbiased with respect to the former is equally likely to yield all possible outcomes. This is evident from the definition above, since if the system is known to be in state $|e_i\rangle$, the probability that it will be found in the state $|f_j\rangle$ is given by $\left|\langle f_j | e_i \rangle\right|^2$, which is defined to be equal for all possible measurements. Note that this definition is not limited to only two bases; it is completely possible to find sets of three or more bases that are all mutually unbiased with respect to eachother.

Since MU bases are important in many applications of quantum theory, it is naturally of great interest to find sets of bases with this property, but this task has proven to be hard. It has been shown that the maximum number of bases that can be MU with respect to each other is no more than $d + 1$ for a Hilbert space $\mathbb{C}^d$ of dimension d. Such a set of $d + 1$ MU bases is called a *complete set*. At the present, it is not clear whether all Hilbert spaces has a complete set of MU bases. It has, however, been proven that a complete set of MU bases exists if the dimension d is an integral power of a prime number, but it is still an open question whether non-prime power-dimensional Hilbert spaces may have a complete set of MU bases.[10]

The smallest dimension for which the maximum number of MU bases it not currently known is when $d = 6$. Searches for MU bases in six dimensions have so far only found sets of three such bases, and it has been conjectured that this is the maximum number. Moreover, it has been suggested that this should be the case for any dimension $d = 2(2n + 1), n \in \mathbb{Z}$, i.e. any odd number multiplied by two.

The concept of MU bases can be generalized to infinite-dimensional continuous variable Hilbert spaces. It is then only natural to pose the same question for these Hilbert spaces; what is the maximum number of MU bases that can be found in an infinite-dimensional Hilbert space? This subject has become somewhat of a controversy, where some researchers hold that these Hilbert spaces must behave characteristically like prime-dimensional spaces and have an infinite number of MU bases, whilst others believe they are limited to only three such bases, as is conjectured for some non-prime finite-dimensional spaces.

Further understanding of MU bases in continuous variable Hilbert spaces may provide valuable insights to the nature of MU bases in finite-dimensional Hilbert spaces. Additionally, continuous variable MU bases have analogous applications to the finite-dimensional case, which makes them a powerful tool and an important field of study [4]. In this report I will present an investigation into continuous variable MU bases, with the intent of shedding some light on the questions raised above. Specifically, I will examine the results of Weigert and Wilkinson [5], which suggest that the maximum number of MU bases in a continous variable Hilbert space is three, and give my take on the controversy surrounding this subject by adressing some of the critique that these results have met [10] with an alternative approach to the problem.

2 Continuous Variable MU Bases

2.1 Overview

There is an interest in generalizing the concept of mutually unbiased bases to infinite dimensional continuous variable spaces, such as the Hilbert space $\mathcal{L}_2$ of square integrable functions. This Hilbert space is spanned by the eigenstates of the position and momentum operators, $\hat{q}$ and $\hat{p}$ respectively[1], which are known to satisfy the condition

$$|\langle q|p\rangle|^2 = \frac{1}{2\pi\hbar} \tag{2.1}$$

Weigert and Wilkinson thus suggest in [5] that a natural generalization of MU bases to continuous variables take on the form

$$\left|\left\langle \Psi_s \middle| \Psi'_{s'} \right\rangle\right| = k > 0 \qquad \forall\, s, s' \in \mathbb{R},\;\; k \in \mathbb{R} \tag{2.2}$$

for orthonormal bases $\{|\Psi_s\rangle\}_{s\in\mathbb{R}}$. With this definition, the eigenbases of momentum and position are MU with respect to each other.

Since the momentum and position bases are related to each other via a rotation in phase space, it is reasonable to believe that a linear combination of the two might constitute a third MU basis. The eigenstates $|q_\theta\rangle$ of the operator $\hat{q}_\theta = \cos\theta\hat{q} + \sin\theta\hat{p}$ are stated in [5] to satisfy the condition 2.2, with

$$|\langle q_\theta|q\rangle|^2 = \frac{1}{2\pi\hbar\,|\sin\theta|} \quad \text{or, equivalently} \quad |\langle q_\theta|q_{\theta'}\rangle|^2 = \frac{1}{2\pi\hbar\,|\sin(\theta-\theta')|} \tag{2.3}$$

This result will be proven later (section 2.2.4). This *state overlap* depends only on the relative angle θ, which makes it possible to construct a set of *at most* three bases that are mutually unbiased with respect to each other. This is because of the fact that one can only find three angles whose pairwise difference is equal. This type of set is named suitably as a *symmetric triple*. An example of such a set is the one composed of the bases $\mathcal{B}_\pm = \{|q_\pm\rangle\}$ with $\hat{q}_\pm = \cos(2\pi/3)\hat{q} \pm \sin(2\pi/3)\hat{p}$ and $\mathcal{B}_q = \{|q\rangle\}$.

Weigert and Wilkinson also note that it's possible to set up a second type of MU triple that is *not* symmetric. This can be done by complementing the bases $\mathcal{B}_q$ and $\mathcal{B}_p$ with the eigenbasis of the operator $\hat{q} - \hat{p}$, which can be obtained by multiplying the operator $\hat{q}_{-\pi/4}$ by $\sqrt{2}$, which compensates for the fact that the bases are not symmetric. The resulting set of bases is named an *asymmetric* triple. The following sections will give a closer look at these results.

[1] While position is more commonly labeled by an x, in this text the alternative notation of q is used; which is usually used when the variable may have interpretations other than spatial position, such as field quadrature.

2.2 State overlap via Wigner functions

2.2.1 Wigner functions

To prove the state overlap formula, one needs to apply the use of Wigner functions. The Wigner function is a quantum phase space *quasiprobability* distribution that can be used as an alternative representation of a quantum state [6][7]. It is a complete description of a quantum state, in the sense that knowing the Wigner function of a system is equivalent to knowing its density matrix. It is defined as[2]

$$W_\Psi(q,p) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ipx}\left\langle q - \frac{x}{2}\middle|\Psi\right\rangle\left\langle\Psi\middle|q + \frac{x}{2}\right\rangle dx \tag{2.4}$$

This alternative representation of a quantum state has valuable properties, such as containing the position and momentum probability distributions in it's marginals:

$$\int_{-\infty}^{\infty} W_\Psi(q,p)dp = |\Psi(q)|^2 \quad \text{and} \quad \int_{-\infty}^{\infty} W_\Psi(q,p)dq = |\Psi(p)|^2 \tag{2.5}$$

It should be noted that a true phase space probability distribution is not achievable in quantum mechanics, since position and momentum are conjugate variables and cannot be fully determined at the same time. The Wigner function resembles a probability distribution in some regards, but it's not a distribution of probability. The Wigner function can, for example, be negative in some regions, whilst a true probability distribution is required to be positive everywhere. For these reasons, the Wigner function is regarded as a quasiprobability distribution. It's important to note that, while the Wigner function itself is not a real probability distribution, it does reproduce the probability distributions over q and p, and these are real probability distributions.

The Wigner function representation is useful when working with continuous variable MU bases because it provides a convenient way to calculate the overlap between two states:

$$|\langle\Psi_1|\Psi_2\rangle|^2 = 2\pi\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_1(q,p)W_2(q,p)dqdp \tag{2.6}$$

[2] Units are here chosen so that $\hbar = 1$. This convention will be employed in the rest of the text.

2.2.2 Position eigenstate Wigner function

To prove equation 2.3 using the Wigner overlap formula 2.6 requires knowing the Wigner functions of the states. It is easiest to begin by finding the Wigner function of position eigenstates, as follows:[3]

$$\begin{aligned}
W_Q(q,p) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ipx}\left\langle q-\frac{x}{2}\middle|Q\right\rangle\left\langle Q\middle|q+\frac{x}{2}\right\rangle dx \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ipx}\delta(q-\frac{x}{2}-Q)\delta(Q-q-\frac{x}{2})dx \\
\left[u = Q-q-\frac{x}{2}\right] \qquad &= \frac{1}{\pi}\int_{-\infty}^{\infty} e^{2ip(u+q-Q)}\delta(u+2q-2Q)\delta(u)du \\
&= \frac{1}{\pi}e^{2ip(q-Q)}\delta(2q-2Q) \;\equiv\; \frac{1}{2\pi}\delta(q-Q)
\end{aligned} \tag{2.7}$$

In the last step, the complex exponential may be discarded since it equals unity when $\delta(q-Q) \neq 0$.

Figure 2.1 to the right illustrates this Wigner function. It is a delta function in phase space, meaning it is zero everywhere except those points where $q = Q$, where it is infinite. This is illustrated in the figure as a contour line where the function is non-zero.

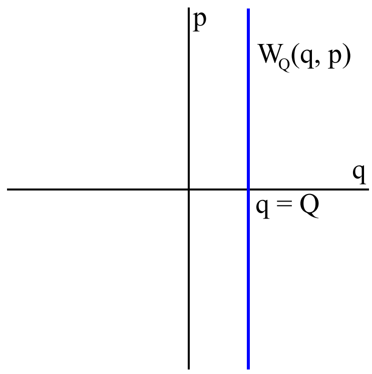

Figure 2.1: A Wigner function corresponding to a position eigenstate.

[3]When dealing with Wigner functions, the eigenvalue of the state both labels the function and occurs in it, so for clarity the eigenvalues are given by a capital letter.

2.2.3 Rotated eigenstate Wigner function

The Wigner function representation of the eigenstates of $\hat{q}_\theta$ may be found via a rotational transformation of the position eigenstate function in phase space. The operator $\hat{q}_\theta$ is given by

$$\hat{q}_\theta = \hat{U}^\dagger(\theta)\hat{q}\hat{U}(\theta) \tag{2.8}$$

where $\hat{U}(\theta) = \exp(-i\theta\hat{a}^\dagger\hat{a})$ is the phase shift operator. This implies that

$$\hat{U}^\dagger(\theta)\,|q\rangle = |q_\theta\rangle \tag{2.9}$$

Since

$$\hat{U}^\dagger(\theta)\hat{q}\hat{U}(\theta)\,|q_\theta\rangle = q_\theta\,|q_\theta\rangle \tag{2.10}$$

and by multiplying both sides from the left with $\hat{U}(\theta)$ and using the unitarity property of the phase shift operator,

$$\hat{q}\hat{U}(\theta)\,|q_\theta\rangle = q_\theta\hat{U}(\theta)\,|q_\theta\rangle \tag{2.11}$$

In other words, $\hat{U}(\theta)\,|q_\theta\rangle$ is an eigenstate of $\hat{q}$ and, equivalently, $\hat{U}^\dagger(\theta)\,|q\rangle$ is an eigenstate of $\hat{q}_\theta$.

Phase shifting a quantum state can be interpreted as rotating it in phase space. One of the basic postulates from which the Wigner function representation *is derived* is that after phase shifting the quantum state, the phase space distribution should rotate accordingly. This is done by demanding that the marginal distributions rotate with the state, yielding new distributions given by:

$$\begin{aligned} |\Psi_\theta(q)|^2 = \int_{-\infty}^{\infty} W_{\Psi_\theta}(q,p)dp &= \langle q|\Psi_\theta\rangle\,\langle\Psi_\theta|q\rangle = \langle q|\,U^\dagger(\theta)\,|\Psi\rangle\,\langle\Psi|\,U(\theta)\,|q\rangle \\ &= \int_{-\infty}^{\infty} W_\Psi(\cos(\theta)q + \sin(\theta)p, \cos(\theta)p - \sin(\theta)q)dp \end{aligned} \tag{2.12}$$

In other words, the Wigner function of a phase shifted state is given by rotating the original Wigner function in phase space.[4] Thus, the Wigner functions of the $\hat{q}_\theta$ eigenstates can be found by a phase space rotation of the position eigenstate Wigner function:

$$W_{Q_\theta}(q,p) = \frac{1}{2\pi}\delta(\cos\theta q + \sin\theta p - Q_\theta) \tag{2.13}$$

An example of this Wigner function is illustrated in figure 3.1 to the right. It is a delta-function whose orientation in phase space is given by θ, and it's displacement from the origin is determined by the eigenvalue Q_θ.

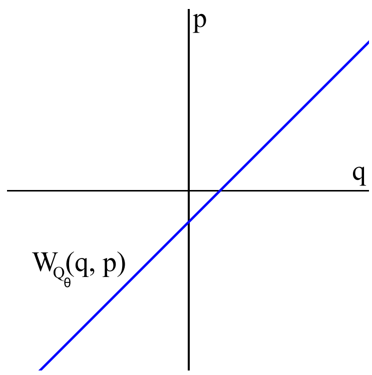

Figure 2.2: A Wigner function corresponding to a rotated eigenstate.

[4] Interestingly, the marginal distributions alone are not enough to characterize a state, but given sufficiently many phase shifted distributions it is possible to experimentally reconstruct the Wigner function to desired accuracy, and thereby also the density matrix. See for example [6] and [9]

2.2.4 Proof of overlap formula

Inserting functions 2.7 and 2.13 into equation 2.6 now allows the overlap to be calculated:

$$\begin{aligned}
|\langle q_\theta | q \rangle|^2 &= 2\pi \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_Q(q,p) W_{Q_\theta}(q,p) dq dp \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(q - Q)\delta(\cos\theta q + \sin\theta p - Q_\theta) dq dp \\
&= \frac{1}{2\pi} \int \delta(\cos(\theta)Q + \sin(\theta)p - Q_\theta) dp \\
\left[u = \sin(\theta)p + \cos(\theta)Q - Q_\theta \right] &= \frac{1}{2\pi} \int \delta(u) \frac{du}{|\sin(\theta)|} = \frac{1}{2\pi\,|\sin(\theta)|}
\end{aligned} \tag{2.14}$$

Two things about this result is worth a remark. Firstly, the overlap does not depend on Q or Q_θ, i.e. which states one selects from the bases. This means that these bases are mutually unbiased, according to the definition of continuous variable MU bases 2.2.

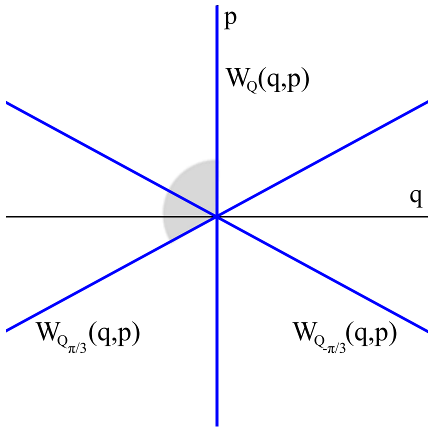

Figure 2.3: The symmetric configuration of MU bases. The gray area signifies an angle of $\pi/3$, the phase angle difference between the states.

Secondly, the overlap depends on the relative phase factor $|\sin\theta|$. This implies that one can't find more than three bases that are all mutually unbiased with respect to eachother, which can be understood in a geometric fashion from the Wigner formulation of the states (illustrated in the figure to the right). The angle θ is the relative phase angle between two states, and it is not possible to construct more than three states whose relative angles are all the same. One can think of the states as being given by vectors pointing in the direction of their orientation in phase space, and it's not possible to find more than three vectors whose pairwise relative angles are all equal (in two dimensions).

This seems to imply that the maximum number of MU bases in a continuous variable Hilbert space is limited to three. However, this result has been met with some skepticism, as will be discussed in section 3.

2.3 Generalized overlap formula

If one allows operators such as $\hat{q} - \hat{p}$, the overlap formula 2.3 can no longer be applied. These operators are given from the rotated position operator $\hat{q}_\theta$ via a simple scalar factor. For example,

$$\hat{q} - \hat{p} = \sqrt{2}\hat{q}_{-\pi/4} \tag{2.15}$$

To find the overlap between eigenstates of such operators, their Wigner functions needs to be known. These can be found by simply noting that the operators are related via a linear transform in position space,

$$\begin{aligned} q \quad &\mapsto \quad q' = \frac{1}{r}q \quad &\iff \quad q = rq' \\ \hat{q} = q \quad &\mapsto \quad \hat{q}' = q' = \frac{q}{r} \quad &\iff \quad \hat{q} = r\hat{q}' \end{aligned} \tag{2.16}$$

The eigenstates of the operator are given by the same transformation, and so the Wigner functions may also be found. One needs only apply the above transformation and repeat the steps of equation 2.7 to find the rescaled Wigner function of position eigenstates:

$$W_{rq}(q,p) = \frac{1}{2\pi}\delta(r(q-Q)) = \frac{1}{2\pi r}\delta(q-Q) \tag{2.17}$$

This Wigner function may then be rotated accordingly, as previously, to yield the desired result. Using this, the overlap formula 2.3 may be corrected for arbitrary linear combinations of the operators $\hat{q}$ and $\hat{p}$:

$$|\langle rq_\theta | q \rangle|^2 = \frac{1}{2\pi r\hbar\,|\sin\theta|} \tag{2.18}$$

With this compensating factor r, it becomes possible to construct the previously mentioned asymmetric triples of bases. This is done by first choosing two bases arbitrarily, without a compensating factor ($r = 1$). It is then possible to find a third base whose pairwise overlaps with the first two are equal, but not necessarily equal to the pairwise overlap *between* the first two. One can then rescale this overlap by setting r so that all three pairwise overlaps are equal.

This is the case with the bases $\mathcal{B}_q$, $\mathcal{B}_p$ and $\mathcal{B}_{q-p}$, where the latter has a compensating factor of $\sqrt{2}$ to make it mutually unbiased with the former two [8]. This situation is illustrated in the figure to the right. However, as will be shown later, this type of configuration of MU bases is not as asymmetric as it may seem (section 4).

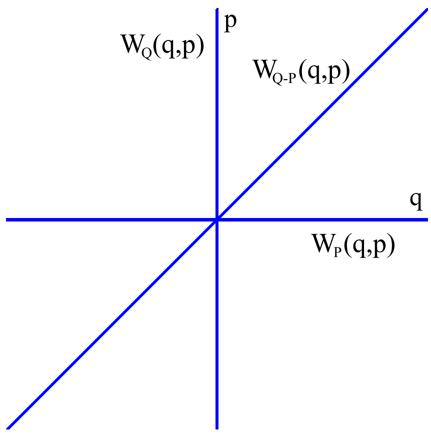

Figure 2.4: The asymmetric configuration of MU bases.

3 Realization by squeezed states

Given definition 2.2, bases of this type are certainly mutually unbiased in the mathematical sense, but some authors have been concerned whether there is any actual physical meaning to these results [10]. In the finite-dimensional definition of mutually unbiased bases 1.4, the state overlap has a direct physical interpretation as a probability, but this is not true for the generalization to continuous variables 2.2. In the continuous variable case, the basis states are idealized and not actually physically realizable states; there is no sense in speaking of a particle having a precise location or momentum. Quantities such as $|\langle q|p\rangle|^2$ have a numerical value but do not represent an actual probability. This naturally leads to the question: is there any meaning in phase shifting these states to achieve different overlaps, or, in other words, does it matter what the value of the overlap is? If not, then the condition that these bases must be configured symmetrically (or asymmetrically) to be MU is not meaningful.

While the eigenstates of position and momentum may not be physically realizable, it is however possible to approximate them with physically realizable states, for which the overlaps do have a physical meaning. A particle may have arbitrarily small variance in its position at the expense of a large variance in momentum (or vice versa), and, if the variance is small enough, the state may closely resemble that of the idealized position or momentum eigenstate.

There is a class of states known as squeezed coherent states (or plainly squeezed states) which are excellent candidates for approximation, for a few reasons. Squeezed states are the most general type of states that have a strictly positive Wigner function, they are states of minimal uncertainty, i.e they obey $\Delta q \Delta p = \frac{1}{2}$, and both their Wigner functions and wave functions are gaussian, so they should be possible to manipulate to closely resemble the desired generalized eigenstates. Furthermore, these states can be created and manipulated in a laboratory environment [11] [12].

Squeezed states are a generalization of coherent states, which are eigenstates of the annihilation operator:

$$\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle \tag{3.1}$$

The most basic such state is the ground state of the harmonic oscillator, since

$$\hat{a}\,|0\rangle = 0 \cdot |0\rangle = 0 \tag{3.2}$$

An arbitrary coherent state can be produced from the vacuum state via a displacement:

$$D(\alpha)\,|0\rangle = |\alpha\rangle \tag{3.3}$$

Where $D(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$ is the displacement operator, which displaces the expectation values of position and momentum by the real and imaginary parts of α, respectively; corresponding to a translation of the wigner function in phase space.

Coherent states have the lowest possible variances in both q and p, with $\Delta q = \Delta p = \frac{1}{\sqrt{2}}$. Squeezed states are the result of taking a coherent state and allowing one of the uncertainties to become smaller at the cost of the other one becoming larger, effectively 'squeezing' the uncertainties under the constraint that their product remains minimal.

The position-space wave function of a non-displaced squeezed state is, as demonstrated by Pauli,

$$\Psi_S(q) = (2\pi\Delta^2 q)^{-1/4} \exp\left[-\frac{q^2}{4\Delta^2 q}\right] \tag{3.4}$$

Note that in the limit where $\Delta^2 q \to 0$, the wavefunction approaches the delta function, which is precisely an eigenstate of position; so for a 'sufficiently' squeezed state, this approximation should become more or less exact.[5] The idea, then, is to let these states act as basis states, and see if they obey the same property of mutual unbiasedness 2.3 as demonstrated for the generalized eigenstates of position and momentum.

3.1 Squeezed state Wigner function

The squeezed state, as mentioned above, satisfies the condition

$$\Delta q \Delta p = \frac{1}{2} \tag{3.5}$$

A natural parametrization of the uncertainties is

$$\Delta q = \frac{1}{\sqrt{2}} e^{-\xi} \qquad \& \qquad \Delta p = \frac{1}{\sqrt{2}} e^{\xi} \tag{3.6}$$

Where ξ is known as the *squeezing parameter*. The wave function can thus be written on the parametric form

$$\Psi_S(q) = (\pi e^{-2\xi})^{-1/4} \exp\left[-\frac{q^2}{2e^{-2\xi}}\right] \tag{3.7}$$

To find the Wigner function representation of the squeezed state, the wave function is plugged into equation 2.4:

$$W_S(q,p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ipx} \Psi_S(q - \frac{x}{2}) \Psi_S^*(q + \frac{x}{2}) dx \tag{3.8}$$

The integral may then be evaluated in a few steps:

$$\begin{aligned}
W_S(q,p) &= \frac{1}{2\pi} \frac{1}{\sqrt{\pi e^{-2\xi}}} \int_{-\infty}^{\infty} e^{ipx} \exp\left[-\frac{(q-\frac{x}{2})^2}{2e^{-2\xi}}\right] \cdot \exp\left[-\frac{(q+\frac{x}{2})^2}{2e^{-2\xi}}\right] dx \\
&= \frac{1}{2\pi} \frac{1}{\sqrt{\pi e^{-2\xi}}} \int_{-\infty}^{\infty} e^{ipx} \exp\left[-\frac{q^2 + \frac{x^2}{4}}{e^{-2\xi}}\right] dx \\
&= \frac{1}{2\pi} \frac{1}{\sqrt{\pi e^{-2\xi}}} \exp\left[-e^{2\xi} q^2\right] \int_{-\infty}^{\infty} e^{ipx} \exp\left[-\frac{e^{2\xi} x^2}{4}\right] dx \\
\left[x = -2\pi u\right] &= \frac{1}{\sqrt{\pi e^{-2\xi}}} \exp\left[-e^{2\xi} q^2\right] \int_{-\infty}^{\infty} e^{-2\pi i p u} \exp\left[-\pi^2 e^{2\xi} u^2\right] du
\end{aligned} \tag{3.9}$$

The integral on the last line is simply the fourier transform of a gaussian function. The result of this integral is another gaussian function in p, and the Wigner function becomes

$$W_S(q,p) = \frac{1}{\pi} \exp\left[-e^{2\xi} q^2 - e^{-2\xi} p^2\right] \tag{3.10}$$

[5] See Discussion Section.

This is of course not the most general form of a squeezed state Wigner function, but it can easily be produced from the above. Similar to how phase shifting a state constitutes a rotation in phase space, it can be shown ([6]) that displacing the state is equivalent to translating the Wigner function in phase-space, i.e. applying the displacement operator $D(\alpha)$ to the wave function $\Psi_S(q)$ results in the following translation:

$$W(q,p) \mapsto W'(q',p') = W(q'-q_0, p'-p_0) \qquad q_0 = \Re(\alpha) \ \& \ p_0 = \Im(\alpha) \tag{3.11}$$

An arbitrary squeezed state can thus be produced by rotating and translating the above Wigner function in phase space.

The Wigner function of a squeezed state is a gaussian function with level curves of elliptic shape, as illustrated in the figure to the right. The function in blue ($W_{S_1}(q,p)$) corresponds to a squeezed state that has a positive squeeze parameter, so that it's variance in q is low and it's variance in p is high. One can see that the squeezing of the variances has a direct geometric meaning in phase space, it effectively squeezes the ellipse. In this way, the variances of a state can be deduced directly from it's phase space diagram. The function in red corresponds to a squeezed state that also has been phase shifted and displaced.

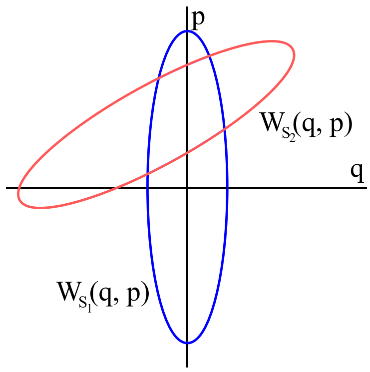

Figure 3.1: Example contour-line diagram of squeezed state Wigner functions.

3.2 Squeezed state overlap

Using the Wigner function 3.10 derived above, the overlap of two squeezed states can be calculated from the overlap formula 2.6. Given two arbitrary such Wigner functions, it is always possible to simplify things by applying a change of coordinates so that one of the states becomes centered at the origin; this does not change the overlap. The problem can be further simplified by rotating the entire setup until the other state is axis-aligned, so that it has no phase dependence but has been displaced. In addition, it will be assumed here that the states have the same squeezing parameter. Thereby, the overlap only depends on the *relative* displacement and phase shift of the two states, and the squeezing parameter. The overlap is then given by:

$$|\langle S_1|S_2\rangle|^2 = 2\pi \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_{S_1}(q,p)W_{S_2}(q,p)dqdp \tag{3.12}$$

Where $W_{S_1}(q,p)$ and $W_{S_2}(q,p)$ are given from 3.10 via the transformations 2.12 and 3.11, respectively, resulting in:

$$|\langle S_1|S_2\rangle|^2 = 2\pi \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_S(\cos(\theta)q+\sin(\theta)p, \cos(\theta)p-\sin(\theta)q)W_S(q-q_0,p-p_0)dqdp \tag{3.13}$$

$$= \frac{2}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left(-e^{2\xi}\left((\cos(\theta)q+\sin(\theta)p)^2+(q-q_0)^2\right)\right) \cdot \exp\left(-e^{-2\xi}\left((\cos(\theta)p-\sin(\theta)q)^2+(p-p_0)^2\right)\right)dqdp \tag{3.14}$$

This integral may be simplified by expanding the squares and writing the resulting quadratic function as a matrix and vector multiplication - the result is a gaussian integral that is easy to solve:

$$|\langle S_1|S_2\rangle|^2 = \frac{2}{\pi}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\bar{q}^T A\bar{q} + \bar{B}^T\bar{q} - C\right) dqdp = \sqrt{\frac{16}{\det A}} \exp\left(\frac{1}{2}\bar{B}^T A^{-1}\bar{B} - C\right) \tag{3.15}$$

Where A a matrix, B is a vector and C is a scalar:

$$A = 2\cdot\begin{pmatrix} e^{2\xi}(1+\cos^2(\theta)) + e^{-2\xi}\sin^2(\theta) & 2\sinh(2\xi)\cos(\theta)\sin(\theta) \\ 2\sinh(2\xi)\cos(\theta)\sin(\theta) & e^{-2\xi}(1+\cos^2(\theta)) + e^{2\xi}\sin^2(\theta) \end{pmatrix} \tag{3.16}$$

$$B = 2\begin{pmatrix} e^{2\xi}q_0 \\ e^{-2\xi}p_0 \end{pmatrix} \qquad C = e^{2\xi}q_0^2 + e^{-2\xi}p_0^2 \tag{3.17}$$

The linear and constant term, given by $\bar{B}$ and C, describe the overlap's dependence on the relative displacement. If one sets $q_0 = p_0 = 0$, these vanish completely and the overlap only depends on the matrix A, as the exponential factor of the overlap then equals unity. It is possible to show that this dependence also vanishes in the limit when $\xi \to \infty$. The claim is that

$$\lim_{\xi\to\infty} \exp\left(\frac{1}{2}\bar{B}^T A^{-1}\bar{B} - C\right) = 1 \tag{3.18}$$

The proof for this is given in Appendix A. This result means that, for sufficiently large values of ξ, the overlap does not depend on how the states have been displaced[6]. This is precisely the same behaviour as that of the generalized eigenstates of position and momentum, whose overlap does not depend on their displacement. The displacement factor can thus be disregarded completely, since ξ can be assumed to be large enough for the approximation to hold.

The overlap is then only a function of the determinant of the matrix A, which, after some simplification, evaluates to:

$$\det(A) = 16(\sin^2(\theta)\sinh^2(2\xi) + 1) \tag{3.19}$$

The end result is

$$|\langle S_1|S_2\rangle|^2 = \sqrt{\frac{1}{\sin^2(\theta)\sinh^2(2\xi) + 1}} \tag{3.20}$$

As can be seen, the overlap of these states have the same inverse dependence on the sine of their relative phase as the generalized eigenstates of position and momentum, which means that the same types of symmetric triples of MU bases could be constructed using these states as a basis. This result indicates that the requirement that the bases are configured in a symmetric fashion is indeed meaningful, and that the value of the overlap does matter, since the generalized eigenstates can be seen as a limiting case of squeezed states when $\xi \to \infty$, for which the overlap has a direct physical interpretation as a probability. As will turn out, asymmetric triples are also possible if one allows the states to have different squeeze parameters.

[6] That is, the displacement dependence can always be made smaller than experimental precision by choice of sufficiently large ξ.

4 Transformational Relation

4.1 Generalized eigenstates

Weigert and Wilkinson present two different possible triples of mutually unbiased bases in [5]. One is the symmetric triple, in which the position eigenbasis is rotated in phase space to yield three mutually unbiased bases, all of which are eigenbases of the operator $\hat{q}_\theta = \cos\theta\hat{q} + \sin\theta\hat{p}$ and are spaced by equal angles θ. The other is the asymmetric triple, in which all of the bases are not spaced by equal angles, but one of the bases have been rescaled to correct for this.

It can be shown that these two basis triples are in fact related via a canonical transformation which preserves overlaps. Given the Wigner function of a state, one can apply a linear phase space transformation to yield a new state:

$$\begin{pmatrix} q \\ p \end{pmatrix} \mapsto \begin{pmatrix} q' \\ p' \end{pmatrix} = T \begin{pmatrix} q \\ p \end{pmatrix}, \quad \det(T) = 1 \quad \Longrightarrow \quad W(q,p) \mapsto W'(q',p') \tag{4.1}$$

If all states have equal overlaps and are subjected to the same linear transformation, the resulting states will also have equal overlaps. This is evident from the integral formulation 2.6 of the overlap,

$$2\pi \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_1(q,p)W_2(q,p)dqdp = 2\pi \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_1'(q',p')W_2'(q',p')\frac{dq'dp'}{\det(T)} \tag{4.2}$$

The overlap of two states is scaled by the determinant of the transformation; applying the same area-preserving transformation to all states thus preserves the overlap, since $\det(T) = 1$. This means that new sets of mutually unbiased bases may be produced from a known set by means of an area-preserving transformation. The most obvious such transformations are of course a common rotation or translation. These types of transformations are trivial, since they preserve the geometry of the set, thus not producing any useful new configurations.

There is a specific type of area-preserving transformation that has some interesting consequences when applied to the basis states at hand. Specifically, the symmetric configuration of bases can be transformed into the asymmetric configuration via a transformation in phase space analogous to a Lorentz-boost[7]:

$$\begin{pmatrix} q \\ p \end{pmatrix} \mapsto \begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} e^{-\xi} & 0 \\ 0 & e^{\xi} \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} \tag{4.3}$$

The orientations of the Wigner functions of the basis states are given by vectors in phase space. Given two arbitrary vectors that are not orthogonal or parallel, it is always possible to transform them to an orthogonal pair via a Lorentz-boost in some direction.

The idea behind this is illustrated in the figure to the right. The black lines illustrates the flow of points in phase space as a Lorentz-boost along the q-axis is applied.

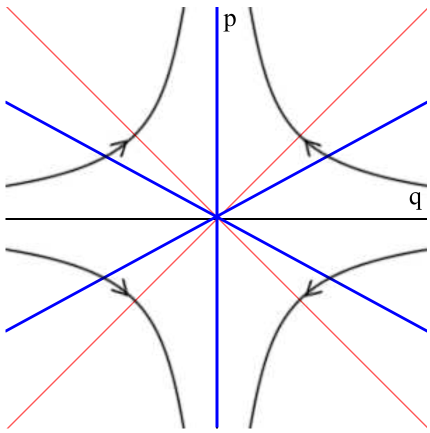

Figure 4.1: Lorentz boost acting on symmetric basis states (blue), transforming them to an asymmetric configuration (red) in phase space.

[7] This is of course not a Lorentz-boost in the same sense as in relativity theory, but the transformation is equivalent.

The symmetric nature of the triple of bases in consideration allows one to orthogonalize two of the vectors without affecting the orientation of the third by choosing the boost axis to be parallel with one of the states.

This diagram makes it reasonable to believe that it should be possible to transform a symmetric triple of bases into an asymmetric one. When the aforementioned transformation 4.18 is applied to the Wigner function of $\hat{q}_\theta$ eigenstates, the result is

$$W'_{Q_\theta}(q', p') = \frac{1}{2\pi}\delta(\cos(\theta)e^{\xi}q' + \sin(\theta)e^{-\xi}p' - Q) \tag{4.4}$$

The new state may be renamed, dropping the '-notation for simplicity,

$$W_{Q_\theta,\xi}(q, p) = W'_{Q_\theta}(q, p) \tag{4.5}$$

To prove that a symmetric triple can be transformed into an asymmetric triple, one may choose any such symmetric set of basis states. For simplicity, it is best to choose the eigenstates of $\hat{q}$, $\hat{q}_{\pi/3}$ and $\hat{q}_{-pi/3}$. The transformed Wigner functions for these states are

$$W_{Q,\xi}(q, p) = \frac{1}{2\pi}\delta(e^{\xi}q - Q) \tag{4.6}$$

$$W_{Q_{\pi/3},\xi}(q, p) = \frac{1}{2\pi}\delta(\frac{e^{\xi}q}{2} + \frac{\sqrt{3}e^{-\xi}p}{2} - Q) \tag{4.7}$$

$$W_{Q_{-\pi/3},\xi}(q, p) = \frac{1}{2\pi}\delta(\frac{e^{\xi}q}{2} - \frac{\sqrt{3}e^{-\xi}p}{2} - Q) \tag{4.8}$$

It is evident that a proper choice of ξ will bring the coefficients in front of q and p to the same values for the second and third states. These states will then be orthogonal (in the sense that their relative angle in phase space is 90°), and it should then be possible to rotate the entire setup so that these become eigenstates of q and p (i.e. align them with the axes). The first state will only be affected in a scaling sense. The sought after value of ξ is thus given by the equation

$$\frac{e^{\xi}}{2} = \frac{\sqrt{3}e^{-\xi}}{2} \iff e^{2\xi} = \sqrt{3} \iff \xi = \frac{\ln(\sqrt{3})}{2} \tag{4.9}$$

The resulting states are:

$$W_{Q,\xi}(q, p) = \frac{1}{12^{1/4}\pi}\delta(\sqrt{2}q - \frac{\sqrt{2}Q}{3^{1/4}}) \tag{4.10}$$

$$W_{Q_{2\pi/3},\xi}(q, p) = \frac{1}{12^{1/4}\pi}\delta(-\frac{1}{\sqrt{2}}q + \frac{1}{\sqrt{2}}p - \frac{\sqrt{2}Q}{3^{1/4}}) \tag{4.11}$$

$$W_{Q_{-2\pi/3},\xi}(q, p) = \frac{1}{12^{1/4}\pi}\delta(-\frac{1}{\sqrt{2}}q - \frac{1}{\sqrt{2}}p - \frac{\sqrt{2}Q}{3^{1/4}}) \tag{4.12}$$

Now, since $\delta(-x) = \delta(x)$, one can simply flip the signs of the last two states and it becomes apparent that this is the asymmetric triple $[\mathcal{B}_q, \mathcal{B}_p, \mathcal{B}_{q-p}]$ rotated in phase space by $-\pi/4$, apart from a scaling factor common to all states (this is trivial since it can always be amended by scaling the entire configuration).

This result shows that the triplets of bases presented by Weigert and Wilkinson as two unique possibilities are in fact both part of the same one-parametric family of bases and, in a sense, the asymmetric and the symmetric triples are the same configuration; since it's only a matter of your choice of coordinates.

4.2 Squeezed states

Does the same transformational relation apply to squeezed states? Firstly, one needs to verify that the Wigner function produced when applying a Lorentz-boost to a squeezed state also constitutes a valid squeezed state. The Wigner function of a squeezed state (not taking into acccount displacement) can be written as

$$W_S(q,p) = \frac{1}{\pi}\exp(-\bar{q}^T A\bar{q}) \tag{4.13}$$

Where $\bar{q}^T A\bar{q}$ is a quadratic form and A is the matrix that defines this form, and thereby the state itself. Note that applying a linear transformation $\bar{q} \mapsto \bar{q}' = U\bar{q}$ maps the matrix A to $A' = U^T AU$.

For a state that has not been phase shifted, the matrix A is given by

$$A = \begin{pmatrix} e^{2\xi} & 0 \\ 0 & e^{-2\xi} \end{pmatrix} \tag{4.14}$$

as derived in section 3.1.[8] This matrix is positive definite and has unit determinant, which is no coincidence; all matrices that fulfill these properties define a squeezed state, as will be shown. Any positive definite matrix M can be diagonalized by an orthogonal matrix Q, and if $\det(M) = 1$, it can be written as

$$M = Q^T DQ \quad \text{where} \quad D = \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix} \tag{4.15}$$

The matrix D is on the same form as A, and the matrix Q can be decomposed into rotations and reflections, due to it being orthogonal, so the matrix M is a rotated and/or reflected squeezed state, and these transformations produce valid squeezed states (reflections are equivalent to rotations when acting on symmetric matrices, and rotations have been shown to be valid transformations).

With this in mind, it becomes apparent that any transformation that preserves the unit determinant and positive definiteness of the state's matrix will produce a valid squeezed state. Any invertible transformation with unit determinant fulfills this condition, since, if A is positive definite, then $A' = U^T AU$ is also positive definite if U is invertible.

Since the Lorentz-boost is an invertible transformation with unit determinant, it can be applied to a squeezed state to produce another squeezed state. The goal is to show that it is possible to transform the symmetric triple's states to the asymmetric triple. This is easiest done by starting from an arbitrary squeezed state, which is done by applying a phase shift to the basic state defined by A, i.e. rotating it in phase space:

$$\begin{pmatrix} q \\ p \end{pmatrix} \mapsto \begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} \tag{4.16}$$

[8] The special case when $\xi = 0$ yields the identity matrix, which is the vacuum state with $\Delta q\Delta p = \frac{1}{2}$; all squeezed states are technically Lorentz-boosted coherent states. The idea here is to show that applying a Lorentz boost *after* phase shifting a state is valid.

The state is then described by the matrix:

$$A' = \begin{pmatrix} \cos^2\theta e^{2\xi} + \sin^2\theta e^{-2\xi} & 2\cos\theta\sin\theta\sinh(2\xi) \\ 2\cos\theta\sin\theta\sinh(2\xi) & e^{2\chi}\sin^2\theta e^{2\xi} + \cos^2\theta e^{-2\xi} \end{pmatrix} \tag{4.17}$$

Now, applying a Lorentz-boost

$$\begin{pmatrix} q' \\ p' \end{pmatrix} \mapsto \begin{pmatrix} q'' \\ p'' \end{pmatrix} = \begin{pmatrix} e^{-\chi} & 0 \\ 0 & e^{\chi} \end{pmatrix} \begin{pmatrix} q' \\ p' \end{pmatrix} \tag{4.18}$$

results in the matrix

$$A'' = \begin{pmatrix} e^{-2\chi}(\cos^2\theta e^{2\xi} + \sin^2\theta e^{-2\xi}) & 2\cos\theta\sin\theta\sinh(2\xi) \\ 2\cos\theta\sin\theta\sinh(-2\xi) & e^{2\chi}(\sin^2\theta e^{2\xi} + \cos^2\theta e^{-2\xi}) \end{pmatrix} \tag{4.19}$$

As can be seen, the transformation only acts on the main diagonal elements of the matrix. The matrix is already symmetric, and both elements on the diagonal are non-zero and strictly positive, which means that it is always possible to choose χ such that the matrix becomes bisymmetric:

$$e^{-2\chi}A_{11} = e^{2\chi}A_{22} \iff \chi = \ln\left(\frac{A_{11}}{A_{22}}\right)/4 \tag{4.20}$$

A bisymmetric matrix is invariant under reflections about the 45° line $p = q$. In other words, applying the transformation

$$\begin{pmatrix} q \\ p \end{pmatrix} \mapsto \begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} \tag{4.21}$$

does not change the matrix. This symmetry corresponds to the fact that such a state has it's axis of symmetry parallell or orthogonal to the 45° line.[9]

It is now possible to show that one can transform a symmetric triple of squeezed states into an asymmetric triple. A symmetric triple is given by the states $W_{\theta=\pm\pi/3}(q,p)$ and $W_{\theta=0}(q,p)$. After applying a Lorentz-boost, they are described by the matrices

$$A_{\theta=\pm\pi/3} = \begin{pmatrix} e^{-2\chi}(\cos^2\frac{\pi}{3}e^{2\xi} + \sin^2\frac{\pi}{3}e^{-2\xi}) & \pm 2\cos\frac{\pi}{3}\sin\frac{\pi}{3}\sinh(2\xi) \\ \pm 2\cos\frac{\pi}{3}\sin\frac{\pi}{3}\sinh(-2\xi) & e^{2\chi}(\sin^2\frac{\pi}{3}e^{2\xi} + \cos^2\frac{\pi}{3}e^{-2\xi}) \end{pmatrix} \tag{4.22}$$

$$A_{\theta=0} = \begin{pmatrix} e^{-2\chi+2\xi} & 0 \\ 0 & e^{2\chi-2\xi} \end{pmatrix} \tag{4.23}$$

For the right choice of χ, the matrices $A_{\theta=\pm\pi/3}$ will become bisymmetric, and thus one of the states is oriented parallell to and the other orthogonal to the 45° axis (the matrices have opposite signs on the off-diagonal elements, so they can't *both* be parallell to it, as that would imply they have identical matrices). The state $W_{\theta=0}(q,p)$ does not change it's orientation but gets a boost to it's squeeze parameter. From this, one can tell that the phase space orientations of these states are identical to the orientations of the generalized states in the previous section, 4.10-4.12.

[9]The fact that all bisymmetric matrices (with unit determinant) define states oriented parallel or orthogonal to the 45° axis can also be realized by phase shifting the matrix A by $\theta = \pm\pi/4$ and recognizing that the resulting matrices can take on the form of any bisymmetric matrix with unit determinant.

As in the previous section, it is now possible to apply a rotation to put these states on the form where two states are axis aligned and the other has been rescaled (by a changed squeeze parameter in this case). The result is a triple of squeezed states that are arranged in the fashion of the asymmetric triple (as defined in section 2.3), and whose overlaps are mutual.

This shows that the transformational relation between the symmetric and asymmetric triples demonstrated in the previous section also applies to squeezed states, in the sense that applying a Lorentz-boost to symmetrically arranged squeezed states can produce asymmetrically arranged squeezed states, whose overlaps are mutual. In this sense, the asymmetric triple also has a physically realizable counterpart of squeezed states, with overlaps that are physically meaningful.

5 Discussion

The first main result of this thesis comes from the approximation of generalized eigenbases by squeezed states. It is shown that the overlap of these states behave identically to the generalized eigenstates for sufficiently large values of the squeezing parameter, in the sense that they only depend on relative phase between states. Regarding the generalized eigenstates as the limit of squeezed states shows us that the value of the overlap between two states is physically meaningful. This also means that the condition that the bases must be arranged as a symmetric or asymmetric triple to be mutually unbiased is meaningful. It also demonstrates a way to implement continuous variable MU bases experimentally, since squeezed states can be created and manipulated in the laboratory.

The second result is the transformational relation between the symmetric and asymmetric triples of bases. This result shows that these two configurations of bases are not unique, and in fact represent the same configuration; one triple can be produced from the other by a simple area-preserving transformation (preserves overlaps), just as it's possible to rotate any such configuration to yield a new triple of bases. In this sense, the only difference between these configurations is a choice of coordinates in phase space.

These results also give strength to the hypothesis that the maximum number of mutually unbiased bases in an infinite-dimensional continuous variable Hilbert space is limited to three (for one pair of continuous variables q and p), as it is shown that the overlap between states only depend on the relative phase shift both for generalized eigenstates and squeezed states, in the limit when $\xi \to \infty$.

A natural next step to take would be to further investigate the approximation of position and momentum eigenstates by squeezed states, and ideally derive the overlap formula 2.3 as a limit of the squeezed state overlap.

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6 Appendix A - Displacement invariance for squeezed state overlaps

In section 3.2, it is claimed that

$$\lim_{\xi\to\infty} \exp\left(\frac{1}{2}\bar{B}^T A^{-1}\bar{B} - C\right) = 1 \tag{6.1}$$

This shows that squeezed state overlaps are invarant under displacements for sufficiently large values of ξ. The proof for this is relatively straight-forward. A, $\bar{B}$ and C are given by:

$$A = 2\cdot\begin{pmatrix} e^{2\xi}(1+\cos^2(\theta)) + e^{-2\xi}\sin^2(\theta) & 2\sinh(2\xi)\cos(\theta)\sin(\theta) \\ 2\sinh(2\xi)\cos(\theta)\sin(\theta) & e^{-2\xi}(1+\cos^2(\theta)) + e^{2\xi}\sin^2(\theta) \end{pmatrix} \tag{6.2}$$

$$B = 2\begin{pmatrix} e^{2\xi}q_0 \\ e^{-2\xi}p_0 \end{pmatrix} \qquad C = e^{2\xi}q_0^2 + e^{-2\xi}p_0^2 \tag{6.3}$$

The problem is equal to showing that

$$\lim_{\xi\to\infty} \frac{1}{2}\bar{B}^T A^{-1}\bar{B} - C = 0 \tag{6.4}$$

One has

$$\frac{1}{2}\bar{B}^T A^{-1}\bar{B} = \frac{1}{2}\Big(A^{-1}_{1,1}B_1^2 + 2\underbrace{A^{-1}_{2,1}}_{=A^{-1}_{1,2}} B_1B_2 + A^{-1}_{2,2}B_2^2\Big) \tag{6.5}$$

If 6.1 is to be true, then

$$\lim_{\xi\to\infty}\left(2e^{4\xi}q_0^2A^{-1}_{1,1} + 4q_0p_0A^{-1}_{2,1} + 2e^{-4\xi}p_0^2A^{-1}_{2,2} - e^{2\xi}q_0^2 - e^{-2\xi}p_0^2\right) = 0 \tag{6.6}$$

This can be split up into three limits:

$$\lim_{\xi\to\infty} q_0^2\left(2e^{4\xi}A^{-1}_{1,1} - e^{2\xi}\right) = 0 \quad \& \quad \lim_{\xi\to\infty} p_0^2\left(2e^{-4\xi}A^{-1}_{2,2} - e^{-2\xi}\right) = 0 \quad \& \quad \lim_{\xi\to\infty} q_0p_0A^{-1}_{2,1} = 0 \tag{6.7}$$

The inverse of A is

$$A^{-1} = \frac{e^{2\xi}}{6e^{4x} + e^{8x} + 1 - (e^{4x}-1)^2\cos(2\theta)}\begin{pmatrix} \cos^2\theta + e^{4x}\sin^2\theta + 1 & -2e^{2x}\cos\theta\sin\theta\sinh(2\xi) \\ -2e^{2x}\cos\theta\sin\theta\sinh(2\xi) & e^{4x}\cos^2\theta + \sin^2\theta + e^{4x} \end{pmatrix} \tag{6.8}$$

Using this, one can verify that the above stated limits are all zero, which is easiest done using software like Mathematica due to the complicated form of the inverse matrix elements.