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Local Energy in Newtonian Gravitation

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Abstract

There is considerable freedom regarding how to define local gravitational energy in Newtonian gravitation. We investigate how to formulate Newtonian gravitation for continuous matter fields as a Lagrangian field theory. By a Noetherian analysis, it is then found that the canonical form of the gravitational energy density of a perfect pressureless fluid is two times the usual potential energy density plus a positive field energy. The potential energy term matches the weak-field limit of a relativistic matter energy.

An older model of energy transfer via the Newtonian gravitational field is revisited. Hermann Bondi and William McCrea showed that internal energy can be transmitted inductively between two gravitating bodies by varying their spherical quadrupole moment in a coordinated way. We derive their results in more detail and conclude that the conditions for such energy transfer are indeed attainable. By means of computer algebra, it is confirmed that, in a special case, the mechanical power exerted across the vacuum of space can be calculated from a divergence-free gravitational energy flux density vector.

It is speculated (by us) that continued study of the non-relativistic limit of general relativity could lead to the discovery of connections between relativistic quasi-local gravitational energy and Newtonian local gravitational energy. We suggest that geometrized Newtonian gravitation, Newton-Cartan theory, is the appropriate formulation to use for such endeavors. The thesis is concluded by a brief summary of the workings of such a geometrical view of Newton's theory.

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Contents

Introduction

Gravitational energy in general relativity is a difficult and complicated subject. No one denies that gravity itself contributes energy to a system, but the localization of such energy is not, in general, very well understood. The consensus seems to be that Einstein's relativity principle excludes a truly local notion of gravitational energy [11, pp. 342–344, 18, pp. 466–468]. There is, however, much debate regarding how a quantity of energy-momentum can be associated with finite regions of spacetime; quasi-local gravitational energy [25].

In Newtonian gravitation, localized gravitational energy comes with its own set of complications. While a differential (i.e. local) energy conservation law can be formulated, the gravitational contribution to such conserved energy can always be redistributed between a form of potential energy, carried by the matter, and a field energy, carried by the gravitational field. Such issues with classical physics are not so worrying in themselves; while accurate for a wide range of phenomena, Newton's theory no longer serves as our base understanding of physical reality. One cannot help but wonder, however, if there is not some connection between the problematic local gravitational energy in the two theories of gravitation. If Newtonian gravitation should be considered an approximation to general relativity in the limits of low velocities and weak fields (as is often claimed), then one would like to know how the Newtonian notion of local gravitational energy arises during this limiting procedure.

Extending Newtonian gravitation to a treatment of general continuous matter fields turns out to be, in some ways, harder than one might naively expect. What is needed is the combination of continuum mechanics with a gravitational field sourced by, and acting on, the matter itself. In chapter 1 of this thesis, the concept of energy in such a theory, here called a *Newtonian field theory*, is explored. In particular, we try to answer the question whether gravitational energy should be considered field energy or potential energy of the matter (or some combination of the two). Continuum mechanics is itself a vast subject, so the discussion is limited to perfect fluids. To properly explain the question, and attempt at an answer, we touch upon electrodynamics, fluid dynamics, general relativity, and Lagrangian field theory.

One case where it is generally agreed that relativistic gravitational energy can be approximately localized is in gravitational waves [11, pp. 342–344, 18, pp. 964–966]. As the waves propagate through spacetime, they carry with them energy, which can be absorbed by, for example, an experimental apparatus at some time and place. Before radiative transfer in general relativity was accepted, it was known that Newtonian gravitation predicts its own form of energy transfer via the vacuum of space, a phenomenon one may call *gravitational inductive energy transfer*. In chapter 2, we study an interesting model formulated by Hermann Bondi and William McCrea, where pulsating gravitating bodies exchange energy via the gravitational field. Apart from introducing our readers to Bondi's and McCrea's ideas, we expand on their work by doing something that they could not back in their days. A computer algebra approach lets us make some practical calculations of the corresponding energy flux density fields; calculations which would have been a considerable effort to complete by hand.

To reconnect with the modern problems of local energy in general relativity, we had planned to extend our classical discussion by studying the non-relativistic limit. Our hopes in such a study were two-fold; perhaps Einstein's theory can say something about the origins of local energy in Newton's theory, but even more exciting would be if this could lead to some small advance towards a better understanding of gravitational energy in general relativity. Chapter 3 is an introduction to Newton-Cartan theory, widely considered to be the proper way of describing Newtonian gravitation as a limit of general relativity. Because of time constraints we never reached any conclusions regarding energy, but we hope that this text can be helpful to students and researchers who wish to follow in our footsteps and beyond.

Chapter 1

Energy in Newtonian Field Theory

There has been interest in the past in defining a local measure of energy in Newtonian gravitation; an energy density of the gravitational field. Notably, James Clerk Maxwell wrote in his 1865 work on electrodynamics [17], that the similarities between gravitational attraction and magnetic repulsion imply an "intrinsic energy" (in his notation)

$$\mathrm{E} = \mathrm{C} - \Sigma \frac{1}{8\pi} \mathrm{R}^2 d\mathrm{V}, \tag{1.1}$$

of the gravitational field (R is the resultant gravitational force on the matter, C is an unknown constant). Maxwell thought of this energy as belonging to some kind of gravitational medium (cf. luminiferous aether), from which the presence of gravitational force then must detract energy by the amount $-\mathrm{R}^2/(8\pi)$. In order for the medium not to have negative energy anywhere, then, it must possess an intrinsic energy density greater than the largest possible value of $\mathrm{R}^2/(8\pi)$ anywhere in the universe. The number C is the integral of this intrinsic energy over all of space; a potentially large amount of energy. Maxwell found this conclusion uncomfortable and abandoned his attempt at explaining gravitation as arising from the action of the surrounding medium.

In our times, field quantities are usually accepted without reference to a medium in which they propagate. A modern day interpretation of Maxwell's result would be to assign the field energy density

$$-\frac{1}{8\pi} \partial_i \phi \partial_i \phi \tag{1.2}$$

to the gravitational field, $-\partial_i \phi$, and not worry too much that it is negative (John Lighton Synge did exactly this in his 1972 paper on Newtonian field theory [24]). As was foreshadowed in the introduction to this thesis, Maxwell's suggestion is just one of endless possibilities for assigning an energy density to Newtonian gravitation. Other contributions have placed such energy in the matter distribution, as potential energy, rather than in the field. As an example of the latter, Hermann Bondi based the derivation of his "Newtonian Poynting vector of the gravitational field" on such an assumption [19, pp. 278–279, 26, p. 433]. As shall be shown in detail in the following sections, there is no real contradiction between these viewpoints. One may still want to ask if there is any sense in which one view of localized gravitational energy is more correct than the others.

Recently, an interesting contribution has come from the philosophy of science. In their 2018 paper [3], Dewar and Weatherall apply a variational approach and Noether's theorem as a means of finding a canonical form for classical gravitational energy. Their analysis is, however, subject to the restriction that the matter field is a fixed background structure, rather than a dynamical field. Inspired by their efforts, the current work seeks to expand upon this idea by applying Noether's theorem to a fully dynamical Newtonian field theory. The hard part turns out to be working out the form of the Lagrangian density for Newtonian gravitation as a field theory. In section 1.4, the insights of Seliger and Whitham (extracted from their 1968 paper on the subject [22]) are applied to this end.

Before any of that, however, the problem should be presented in more detail. Since the reader is most likely familiar with the corresponding freedom in electrostatics, a fitting place to start the current study is in electrodynamics with Poynting's theorem (section 1.1). The quasi-static limit of electrodynamics bears a striking resemblance to Newton's theory and, unsurprisingly, a freedom to redistribute energy appears equally in both theories. Following this, a "Poynting's theorem for gravitation" is derived in section 1.2, which will clearly demonstrate the underdetermined nature of Newtonian gravitational energy density.

1.1 Local energy in electrodynamics

Energy conservation in electrodynamics is, according to John Henry Poynting's famous theorem, given by the equation (in geometric units) [6, ch. 27-3, 10, p. 347, 12, p. 259]

$$\int_\omega \mathrm{d}^3x\, E_i J_i + \frac{1}{8\pi}\partial_t \int_\omega \mathrm{d}^3x\, (E_i E_i + B_i B_i) + \frac{1}{4\pi}\int_{\partial\omega} \mathrm{d}n_i\, \epsilon_{ijk} E_j B_k = 0. \tag{1.3}$$

The first term is the total power exerted on the charges and currents in the region ω. If such electrical work is being done, either the field energy density

$$\varepsilon_{\mathrm{EM}} = \frac{1}{8\pi}\,(E_i E_i + B_i B_i) \tag{1.4}$$

must decrease, or energy must flow in over the boundary $\partial\omega$ with the energy flux density given by the *Poynting vector*;

$$S_i = \frac{1}{4\pi}\epsilon_{ijk} E_j B_k. \tag{1.5}$$

Since the conservation law above holds for any closed region, it may be converted into a local law. At every point, then, there is a balance of energy given by the equation

$$E_i J_i + \dot{\varepsilon}_{\mathrm{EM}} + \partial_i S_i = 0, \tag{1.6}$$

which motivates the view of energy as localized and carried by the electric and magnetic fields per the above energy density.

There has certainly been discussion regarding whether this is a unique and correct form of the energy density and flux vector. To begin with, the energy flux density is only determined up to a curl, so there is some freedom in redirecting the flow of energy with no consequence for energy conservation. For coupling electrodynamics to general relativity it is, however, appropriate to not include such a term [12, pp. 608–610]. Further, the energy density and its flux can be modified in a number of ways involving various derivatives of the fields [6, ch. 27-4]. The consensus seems to be that the forms given above are the simplest and best. It is generally agreed that in electrodynamics, electromagnetic energy is field energy (see for example [12, pp. 606–609]).[1]

Electromagnetic theory becomes more closely related to Newtonian gravitation when looking at the electrostatic limit. By setting the time derivative of the magnetic field to zero, the curl of the electric field is made to vanish (Faraday's law says $\epsilon_{ijk}\partial_j E_k = -\dot{B}_i$) and thus it is, in that case, a scalar potential field;

$$E_i = -\partial_i \phi. \tag{1.7}$$

From Gauss's law, Poisson's equation,

$$\nabla^2 \phi = -4\pi\rho, \tag{1.8}$$

can be derived. This would not have worked out the same in full electrodynamics, where the magnetic vector potential is also in play. The Newtonian gravitational field is governed by almost the same field equation; only the sign in the right hand side should differ between the two theories.[2]

The Poynting vector also simplifies somewhat, giving a simpler form of Poynting's theorem for electrostatics. Replacing the electric field with the gradient of the electric potential in the surface integral in equation (1.3) gives the flux

$$-\frac{1}{4\pi}\int_{\partial\omega} \mathrm{d}n_i\, \epsilon_{ijk}\partial_j\left(\phi B_k\right) + \frac{1}{4\pi}\int_{\partial\omega} \mathrm{d}n_i\, \phi\epsilon_{ijk}B_k. \tag{1.9}$$

The first term here is zero, so the flux density may effectively be chosen to be just the integrand of the second term. Using Ampère's law ($\epsilon_{ijk}\partial_j B_k = \dot{E}_i + 4\pi J_i$), this gives a flux density of

$$-\frac{1}{4\pi}\phi\partial_i\dot{\phi} + \phi J_i. \tag{1.10}$$

Local conservation of energy for electrostatics can therefore be stated as

$$\phi\partial_i J_i + \frac{1}{8\pi}\partial_t\left(\partial_i\phi\partial_i\phi\right) - \frac{1}{4\pi}\partial_i\left(\phi\partial_i\dot{\phi}\right) = 0, \tag{1.11}$$

or as

$$-\phi\dot{\rho} + \frac{1}{8\pi}\partial_t\left(\partial_i\phi\partial_i\phi\right) - \frac{1}{4\pi}\partial_i\left(\phi\partial_i\dot{\phi}\right) = 0 \tag{1.12}$$

using conservation of charge ($\dot{\rho} + \partial_i J_i = 0$). The first term represents the electrical power exerted at a point, the second term is the change in electric field energy, and the last term is the flow of field energy out of said point.

Equipped with Poisson's equation, one can perform integration by parts on the electric field energy;

$$\frac{1}{8\pi}\int_{\omega} \mathrm{d}^3x\, \partial_i\phi\partial_i\phi = \frac{1}{8\pi}\int_{\partial\omega} \mathrm{d}n_i\, \phi\partial_i\phi + \frac{1}{2}\int_{\omega} \mathrm{d}^3x\, \rho\phi. \tag{1.13}$$

The energy has been redistributed from the field into a potential energy of the charge distribution plus a surface term. This potential energy obeys the conservation law

$$-\phi\dot{\rho} + \frac{1}{2}\partial_t\left(\rho\phi\right) + \frac{1}{8\pi}\partial_i\left(\dot{\phi}\partial_i\phi - \phi\partial_i\dot{\phi}\right) = 0, \tag{1.14}$$

[1] Some readers may want to bring up the Wheeler-Feynman direct-interaction formulation of electrodynamics, where there is no need for fields at all. The discussion here applies to standard electrodynamics. How energy works in alternative formulations is unknown to us.

[2] The difference in sign accounts for electrostatics being a theory of attracting opposites whereas in gravitation, likes attract.

where the time derivative of the new surface term has been incorporated into the flux of energy density. In electrodynamics, such redistribution of energy would only apply to the longitudinal Helmholtz component of the electric field.

Assuming a compactly supported charge distribution, doing the integrals of equation (1.13) over all of space and setting the potential to zero "at infinity" makes the surface integral vanish (since the integrand goes as one over the distance cubed, thus shrinking faster than the surface grows during a limiting procedure). The total energy of an isolated system is therefore found to be the same regardless of the choice of localization.

There are two competing conservation laws; one entertaining the view that energy is carried by the field, the other that it is carried by the charge distribution, and they each have a corresponding energy flux density vector. It is clear that these can also be mixed in any proportion, so there is in fact an endless supply of different energy densities and flux density vectors. While the choice may be arbitrary in electrostatics, it could be argued that placing the energy in the field is the correct choice since this is the only natural possibility in electrodynamics.

1.2 Poynting's theorem in Newtonian gravitation

To derive a similar theorem of conservation of energy for Newtonian gravitation, take the rate of change in the kinetic energy of a distribution of dust (i.e. a pressureless perfect fluid) inside ω to be

$$\int_\omega \mathrm{d}^3x\, \dot{\tau} = -\int_\omega \mathrm{d}^3x\, \rho v_i \partial_i \phi - \int_{\partial\omega} \mathrm{d}n_i\, \tau v_i. \tag{1.15}$$

Here, v_i is the flow velocity of the matter distribution ρ, τ is the density of kinetic energy ($\rho v_i v_i/2$), and ϕ is the Newtonian potential. The right hand side is the difference between power exerted by body forces inside ω and the rate of kinetic energy flowing out over the boundary, carried away by the flowing matter. Surface forces have been omitted, as there are no such forces in a perfect fluid. Similarly, there is no change in energy related to changes of pressure.

The power equation above can be rewritten as a sum of terms representing different forms of energy. Performing integration by parts on the first term on the right hand side gives

$$\int_\omega \mathrm{d}^3x\, \dot{\tau} = -\int_{\partial\omega} \mathrm{d}n_i\, \rho\phi v_i + \int_\omega \mathrm{d}^3x\, \phi\partial_i\left(\rho v_i\right) - \int_{\partial\omega} \mathrm{d}n_i\, \tau v_i. \tag{1.16}$$

Assuming the matter flows such as to guarantee local conservation of mass, satisfying the continuity equation

$$\dot{\rho} + \partial_i\left(\rho v_i\right) = 0, \tag{1.17}$$

this may further be rewritten as[3]

$$\int_\omega \mathrm{d}^3x\, \dot{\tau} = -\int_{\partial\omega} \mathrm{d}n_i\left(\tau + \rho\phi\right) v_i - \int_\omega \mathrm{d}^3x\, \phi\dot{\rho}. \tag{1.18}$$

The first term on the right hand side is the total energy carried in or out of the region by the flow of matter itself, but the second term still eludes interpretation.

[3]This very useful method of eliminating the flow velocity in favor of a time derivative of the matter distribution was learned from Bondi, who used it to similar effect in [26, p. 433] and in [19].

As can be expected given the earlier discussion on electrostatics, there is more than one way to unpack the remaining term into a time derivative and a surface integral. Using Poisson's equation,[4]

$$\nabla^2\phi = 4\pi\rho, \tag{1.19}$$

to rewrite the integrand as

$$\phi\dot{\rho} = \frac{1}{4\pi}\phi\partial_i\partial_i\dot{\phi} = -\frac{1}{8\pi}\partial_t\left(\partial_i\phi\partial_i\phi\right) + \frac{1}{4\pi}\partial_i\left(\phi\partial_i\dot{\phi}\right) \tag{1.20}$$

reveals a gravitational field energy density and flux density vector. Just as in the electrostatic case, the field energy agrees (up to a surface integral) with a potential energy of the matter;

$$-\frac{1}{8\pi}\int_\omega \mathrm{d}^3x\,\partial_i\phi\partial_i\phi = \frac{1}{2}\int_\omega \mathrm{d}^3x\,\rho\phi - \frac{1}{8\pi}\int_{\partial\omega}\mathrm{d}n_i\,\phi\partial_i\phi. \tag{1.21}$$

Consequently, a general conservation law can be written down as

$$\begin{aligned}\dot{\tau} + \partial_i\left(\left(\tau+\rho\phi\right)v_i\right) + N\left(\frac{1}{2}\partial_t\left(\rho\phi\right) + \frac{1}{8\pi}\partial_i\left(\phi\partial_i\dot{\phi} - \dot{\phi}\partial_i\phi\right)\right)&\\ + \left(N-1\right)\left(\frac{1}{8\pi}\partial_t\left(\partial_i\phi\partial_i\phi\right) - \frac{1}{4\pi}\partial_i\left(\phi\partial_i\dot{\phi}\right)\right) = 0.&\end{aligned} \tag{1.22}$$

The parameter N hence controls the distribution of gravitational energy between placing it in the field and placing it in the matter. We have not found any meaningful way of rewriting the first two terms of this energy balance equation, so their form will be assumed fixed for the remainder of this thesis.

Choosing $N = 0$ places all of the redistributable energy in the field. In this case, the energy density (excluding kinetic energy) found at a point is

$$-\frac{1}{8\pi}\partial_i\phi\partial_i\phi. \tag{1.23}$$

In accordance with Maxwell's expression for gravitational energy, the sign in Poisson's equation has caused the field energy density to be negative. The flux of energy density associated with placing the energy in the field is, similarly to the electrostatic case,

$$\frac{1}{4\pi}\phi\partial_i\dot{\phi}. \tag{1.24}$$

The corresponding conservation law of course agrees with that derived by Synge [24, p. 381].

The case $N = 1$ regards gravitational energy as a potential energy

$$\frac{1}{2}\rho\phi \tag{1.25}$$

of the matter. The formula above replicates Bondi's flux density vector,

$$\frac{1}{8\pi}\left(\phi\partial_i\dot{\phi} - \dot{\phi}\partial_i\phi\right), \tag{1.26}$$

[4] As promised there is an opposite sign compared to electrostatics.

for this choice. Bondi himself points out that this vector is free of divergence in the vacuum and may, for this reason, be of particular interest when studying inductive transfer of gravitational energy (as in that case there should be no transfer if there is no recipient) [26, p. 434]. Bondi's ideas concerning gravitational inductive energy transfer are studied more closely in chapter 2.

The question that was posed is how the energy *should* be distributed; what is the proper value of N? As has been discussed, the form in electrostatics is fixed by requirements of compatibility with electrodynamics and general relativity. For Newtonian gravitation, it is not so easy to come up with these sorts of argument. There is, however, a hint that can be extracted from a relativistic energy. Before moving on to the main event of this chapter (variational principles and Noether's theorem), it is appropriate to dwell a little on this thought.

1.3 Local energy of the static weak field

One possible avenue for fixing the parameter N is to look for something in general relativity that could possibly correspond to this energy in the non-relativistic limit. There is an interesting candidate in the form of a conserved current of matter energy density, which seems to support the case $N = 2$. For further reading about this general relativistic energy, see the 1985 paper [15] by Lynden-Bell and Katz.

Consider a conserved four-vector;

$$\nabla_\mu J^\mu = 0. \tag{1.27}$$

The corresponding coordinate expression

$$\frac{1}{\sqrt{-g}}\partial_\mu \left(\sqrt{-g} J^\mu\right) = 0, \tag{1.28}$$

(valid for any coordinate frame [18, p. 222]) is indeed a classical conservation law. Hence, one might expect that the Newtonian energy density corresponds to a component of some conserved four-vector. Local conservation of energy-momentum is expressed as

$$\nabla_\mu T^{\mu\alpha} = 0, \tag{1.29}$$

but this is not exactly of the expected form.

If a spacetime admits a Killing field, ξ^α, it is possible to construct a conserved matter energy current, defined as

$$J^\alpha = T^{\alpha\mu}\xi_\mu \tag{1.30}$$

for a distribution of matter with energy-momentum tensor $T^{\alpha\beta}$. That this vector is free of four-divergence can be checked by applying Leibniz's rule, local conservation of energy-momentum and Killing's equation;

$$\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0 \tag{1.31}$$

(solutions to this equation are symmetries of the metric) [11, p. 443]. If there is a timelike Killing field, then the associated conserved current can be considered a sort of current of matter energy.

The density distribution of this matter energy does not in general agree with the usual matter density, but rather suffers a form of "redshift" due to gravity. To see this, place some test matter in the form of a perfect fluid at rest in a static spacetime. Such a test fluid is decomposed as

$$T^{\alpha\beta} = (\rho + p)\, u^\alpha u^\beta + p g^{\alpha\beta}, \tag{1.32}$$

where p is the hydrostatic pressure, $g_{\alpha\beta}$ is the metric tensor and u^α is the four-velocity field of the matter. The current of matter energy works out to

$$J^\alpha = (\rho + p)\, u^\alpha u_m \xi^\mu + p\xi^\alpha. \tag{1.33}$$

In this context, "at rest" means that the flow of matter is parallel to the timelike Killing vector;

$$u^\alpha = k\xi^\alpha, \tag{1.34}$$

where k is a normalizing factor such that $u_\mu u^\mu = -1$. Using this in equation (1.33), the current simplifies to

$$J^\alpha = -\rho \frac{1}{k} u^\alpha. \tag{1.35}$$

The time component of this current, as observed in a local inertial frame following the flow, is an energy density given by

$$\mu = u_\mu J^\mu = \frac{\rho}{k}. \tag{1.36}$$

Calling this energy redshifted in scare quotes is appropriate, as light signals exchanged between flow lines of the Killing field are redshifted by the factor $1/k$.

The Schwarzschild metric reproduces Newtonian trajectories in the weak field limit. To find a non-relativistic limit of the above matter energy, place some test matter at rest outside a non-rotating star or planet. Choosing the usual Schwarzschild coordinates, the timelike Killing field gets the components

$$\xi^\alpha = \delta^{0\alpha}. \tag{1.37}$$

Using the normalizing condition $u_\mu u^\mu = -1$ (with $u^\alpha = k\xi^\alpha$) then gives the equation

$$-k^2 \left(1 - \frac{2M}{r}\right) = -1, \tag{1.38}$$

for the redshift factor. So, the matter energy density is, in this case,

$$\mu = \rho \sqrt{1 - \frac{2M}{r}}. \tag{1.39}$$

In the non-relativistic limit, one makes the identification

$$\phi(r) = -\frac{M}{r} \tag{1.40}$$

and expands μ to first order in "small ϕ" to obtain

$$\mu \approx \rho\,(1 + \phi)\,. \tag{1.41}$$

The result suggests that the non-relativistic energy density of matter at rest is not the rest mass itself, but also includes a correction in the form of potential energy. Interestingly, the potential energy is twice the amount usually considered in Newtonian gravitation. This is perhaps a little disappointing at first sight, but there is a workaround. It is conceivable that any Newtonian field energy has been left out in the limiting procedure shown here. Setting $N = 2$ in equation (1.22) and adding the mass density as a form of local energy, one can come up with the energy density[5] (excluding kinetic energy)

$$\rho + \rho\phi + \frac{1}{8\pi}\partial_i\phi\partial_i\phi \tag{1.42}$$

for Newtonian gravitation. This is the energy received as a limit plus a gravitational field energy. It should perhaps be explained more in detail exactly why this field energy went missing, but the above is about as rigorous as we can make the argument at this time. There is some room for further investigation here.

This choice of energy is interesting with regards to Maxwell's worries about negative gravitational energy. The above field energy is positive while the potential energy is negative. For the weak fields to which this reasoning applies, however, it is safe to assume that the negative potential energy is offset by the positive mass density, making the energy density positive everywhere in the weak field. So, in a way, Maxwell was right. There *is* an intrinsic energy of the "medium"; rest mass. In those parts of the universe where the Newtonian energy density above would be negative, a better theory of gravitation is needed (but Maxwell had no chance of knowing this). It is unclear to us whether this should be considered a case of amazing coincidence or of amazing foresight on the part of Maxwell.

1.4 Newtonian field theory from a variational principle

It is well known that Poisson's equation can be derived from the Lagrangian density

$$\mathscr{L} = -\rho\phi - \frac{1}{8\pi}\partial_i\phi\partial_i\phi. \tag{1.43}$$

For this to work, it is necessary to consider the matter density not as a dynamical field, but rather as a fixed background structure. It is not too hard to see why, as demanding that variations with respect to ρ vanish implies a trivial result for ϕ. It is clear that some more structure is needed in order to formulate a proper gravitational field theory.

Keeping ρ fixed when finding the extremum has consequences for the application of Noether's theorem. The stress-energy tensor found by Dewar and Weatherall (in [3]) is not divergence-free; rather, it is sourced by the matter distribution. Put another way, the energy conservation law they found is not of the form

$$\dot{\varepsilon} + \partial_i S_i = 0, \tag{1.44}$$

which is normally expected. This issue confuses the analysis slightly, as it is not easy to draw any conclusion about the value of N in the general Newtonian conservation law (1.22). Their idea of using the canonical formalism to produce a sort of answer regarding what the energy density of gravitation *should* be is an interesting one, none the less.

To supplement the paper by Dewar and Weatherall, this work seeks to expand the above Lagrangian density into a fully dynamical theory. This is a subject not typically covered in physics textbooks, and finding out "how to" is not trivial even though the problem of course has been solved. Luckily, a 1968 paper by Seliger and Whitham [22] outlines how this can be achieved. They do not couple their theory to gravitation specifically, but they do show how to derive the necessary fluid dynamics from a variational principle. In the following sections, a version of their variational

[5] The attentive reader will complain that the result in Newtonian gravitation was derived for a dust solution, not a general perfect fluid. The result for μ works out the same for dust ($p = 0$), however.

principle is developed with the coupling to the gravitational field included from the start. Once the correct field theory has been formulated, Noether's theorem is applied to find the canonical form of the energy density.

1.4.1 Newtonian field equations and the material derivative

Before getting to the actual variational principle, the rules of a Newtonian field theory should be stated clearly once and for all. In such a theory, there are three fields; the matter distribution ρ, the flow velocity v_i, and the Newtonian potential ϕ. The motion of the matter is governed by *Euler's equation*;[6]

$$\dot{v}_i + v_j \partial_j v_i = g_i \tag{1.45}$$

where g_i is the acceleration of the matter due to body forces. There is also continuity of mass;

$$\dot{\rho} + \partial_i (v_i \rho) = 0. \tag{1.46}$$

This is in contrast with electrodynamics, where continuity of charge is a consequence of how the fields are sourced by charges and currents. Here, mass conservation needs to be added in by hand. Finally, the potential is sourced by the matter according to Poisson's equation;

$$\nabla^2 \phi = 4\pi\rho. \tag{1.47}$$

If there are no other forces than gravity, then the acceleration is just

$$g_i = -\partial_i \phi. \tag{1.48}$$

In this theory there is no stress, nor is there any pressure in the matter; it is dealing with a perfect pressureless liquid. It is of course possible to add in more structure, such as a stress tensor, but the goal here is to get at gravitational energy by setting up the simplest possible gravitational field theory. How to include rigidity constraints, stress, and elasticity could be interesting topics for continued investigation, but that has not been pursued here.

The law of motion, and some other formulae that will feature in the coming sections, can be put in a more elegant and intuitive way by use of the *material derivative*, defined as

$$D_t f(x_i, t) = \dot{f}(x_i, t) + v_j(x_i, t)\partial_j f(x_i, t) \tag{1.49}$$

(where f is any field). Intuitively, the material derivative performs "differentiation following the medium", meaning that it describes the change in a quantity as it is experienced by an element of matter following the flow v_i. To see that this makes sense, imagine that an infinitesimal pebble is dropped in the flowing medium, its position given by $X_i(t)$ at any time t. Evaluating some field quantity along this curve produces a composite function, $f(X_i(t), t)$, of time only. The time derivative of this composite is

$$\frac{\mathrm{d}}{\mathrm{d}t} f(X_i(t), t) = \dot{f}(X_i(t), t) + \dot{X}_j(t)\partial_j f(X_i(t), t). \tag{1.50}$$

[6] Euler's equation usually includes motion due to the pressure of the matter. The equation seen here is the special case for a pressureless liquid.

The quantity $\dot{X}_i$ is the flow velocity itself (the pebble follows a curve such that the flow velocity is tangent to it), so

$$D_t f(x_i, t) = \frac{\mathrm{d}}{\mathrm{d}t} f(X_i(t), t) \tag{1.51}$$

when $X_i(t) = x_i$ (i.e. when the pebble passes through the position x_i). This view of coordinates following the flow around is usually referred to as the *Lagrangian* description, but in this work the *Eulerian* view of evaluating quantities by stationary (with respect to the frame) coordinates is used (as is more common in field theory). Even so, the material derivative is a useful tool whenever dealing with flowing matter.

To convince the reader that the choice of law of motion is sensible, Euler's equation is restated as

$$D_t v_i = -\partial_i \phi. \tag{1.52}$$

Its meaning is now clear; the force on the flowing matter is just the gravitational force.

1.4.2 A Lagrangian density for Newtonian field theory

A glance at the Lagrangian density (1.43) suggests that what is missing is the kinetic energy of the matter, but naively adding it in does not solve the problem. Doing so just produces a trivial result for the flow velocity. Following Seliger and Whitham, then, one draws the conclusion that the variation must be further constrained during optimization. Adding in conservation of mass as a side condition, using a Lagrange multiplier, generates the expected field equations but restricted to the special case of irrotational flows.

Consider the action integral

$$S[\lambda, \rho, \phi, v_i] = \int_{t_I}^{t_F} \mathrm{d}t \int_\Omega \mathrm{d}^3x \left(\frac{1}{2}\rho v_i v_i - \frac{1}{8\pi}\partial_i\phi\partial_i\phi - \rho\phi + \lambda\left(\dot{\rho} + v_i\partial_i\rho + \rho\partial_i v_i\right)\right) \tag{1.53}$$

where the spatial integral is over all space (Ω), while t_I and t_F are some arbitrary initial and final times of the evolution of the system. The variation (with respect to all the fields) of this action is

$$\delta S = \int_{t_I}^{t_F} \mathrm{d}t \int_\Omega \mathrm{d}^3x \Bigg(\partial_t(\lambda\delta\rho) + \partial_i(\lambda v_i \delta\rho) - \frac{1}{4\pi}\partial_i(\partial_i\phi\delta\phi) + \partial_i(\lambda\rho\delta v_i)$$

$$+\delta\lambda\left(\dot{\rho} + \partial_i(\rho v_i)\right) \qquad \text{MASS CONTINUITY EQUATION} \tag{1.54a}$$

$$+\delta\rho\left(\frac{1}{2}v_i v_i - \phi - \dot{\lambda} - v_i\partial_i\lambda\right) \tag{1.54b}$$

$$+\delta\phi\left(\frac{1}{4\pi}\nabla^2\phi - \rho\right) \qquad \text{POISSON'S EQUATION} \tag{1.54c}$$

$$+\delta v_i\left(\rho v_i - \rho\partial_i\lambda\right)\Bigg) \tag{1.54d}$$

after the usual procedure of integrating by parts. To make the surface terms vanish, demand that the fields are held fixed at t_I and t_F (i.e. that the variations are set to zero there) and that they are functions that can be expanded in powers of the inverse distance from the origin. Under such conditions, then, the action will be extremal when the four Euler-Lagrange equations are satisfied, that is, when the expressions in parentheses of each row a–d is set to zero. Two rows have been labeled by the name of the corresponding field equation.

The law of motion is derived from the two nameless rows. The field equation received from row (1.54d) above states that at every point where there is some matter, the velocity field satisfies the condition

$$v_i = \partial_i \lambda, \tag{1.55}$$

(i.e. that it is a potential flow there).[7] From row 1.54b, by taking a spatial partial derivative (∂_i) and using the velocity potential, one finds the equation

$$\dot{v}_i + v_j \partial_j v_i = -\partial_i \phi \tag{1.56}$$

or

$$D_t v_i = -\partial_i \phi. \tag{1.57}$$

This is the wanted special case of Euler's equation.

While this Lagrangian density achieves the stated goals, it would be more satisfying to work with a general flow. There is a way to do this, but in order to understand it, one needs to know about the *Clebsch representation.* Flows that solve Euler's equation can be decomposed as [14, pp. 248–249]

$$v_i = \partial_i \lambda + \alpha \partial_i \beta, \tag{1.58}$$

where λ, α, and β are all scalar fields; the *Clebsch potentials.* In the above Lagrangian density, the Lagrange multiplier λ ends up being interpreted as a Clebsch potential. It seems plausible that adding in more constraints in the Lagrangian density could give the Clebsch representation of a general flow as a field equation.

There is some likelihood of stumbling across the action integral above by just trying things out, but the solution towards achieving a general flow is entirely non-obvious. To introduce the full Clebsch representation, another continuity constraint, for a scalar field κ, is needed. Take the action integral to be

$$\begin{aligned} S\left[\beta, \kappa, \lambda, \rho, \phi, v_i\right] = \int_{t_I}^{t_F} \mathrm{d}t \int_\Omega \mathrm{d}^3 x \Bigg(& \frac{1}{2}\rho v_i v_i - \frac{1}{8\pi}\partial_i \phi \partial_i \phi - \rho\phi \\ & + \lambda\left(\dot{\rho} + v_i \partial_i \rho + \rho \partial_i v_i\right) + \beta\left(\dot{\kappa} + v_i \partial_i \kappa + \kappa \partial_i v_i\right) \Bigg), \end{aligned} \tag{1.59}$$

[7] What the fields v_i and λ are doing in the vacuum seems to not be of any consequence, as long as they are sufficiently smooth everywhere and die out fast enough so that the surface terms in the variation of the action all vanish.

which gives the variation

$$\delta S = \int_{t_I}^{t_F} \mathrm{d}t \int_\Omega \mathrm{d}^3x \Bigg(\partial_t\left(\lambda\delta\rho\right) + \partial_t\left(\beta\delta\kappa\right) - \frac{1}{4\pi}\partial_i\left(\partial_i\phi\delta\phi\right) + \partial_i\left(\lambda v_i\delta\rho\right)$$

$$+\partial_i\left(\lambda\rho\delta v_i\right) + \partial_i\left(\beta\kappa\delta v_i\right) + \partial_i\left(\beta v_i\delta\kappa\right)$$

$$+\delta\beta\left(\dot{\kappa} + \partial_i\left(\kappa v_i\right)\right) \tag{1.60a}$$

$$-\delta\kappa\left(\dot{\beta} + v_i\partial_i\beta\right) \tag{1.60b}$$

$$+\delta\lambda\left(\dot{\rho} + \partial_i\left(\rho v_i\right)\right) \qquad \textsc{Mass Continuity Equation} \tag{1.60c}$$

$$+\delta\rho\left(\frac{1}{2}v_i v_i - \phi - v_i\partial_i\lambda - \dot{\lambda}\right) \qquad \textsc{Euler's Equation}^{8} \tag{1.60d}$$

$$+\delta\phi\left(\frac{1}{4\pi}\nabla^2\phi - \rho\right) \qquad \textsc{Poisson's Equation} \tag{1.60e}$$

$$+\delta v_i\left(\rho v_i - \rho\partial_i\lambda - \kappa\partial_i\beta\right)\Bigg). \qquad \textsc{Clebsch Representation} \tag{1.60f}$$

The field equation derived from row f states that the flow velocity satisfies the Clebsch representation

$$v_i = \partial_i\lambda + \frac{\kappa}{\rho}\partial_i\beta, \tag{1.61}$$

wherever $\rho \neq 0$. The material derivative $D_t v_i$ is found from the equations implied by rows a to d. This, again, gives

$$D_t v_i = -\partial_i\phi, \tag{1.62}$$

but now v_i can include vortices as per the new toroidal term in the Clebsch representation. As was promised, requiring "conservation of κ"[9] has given the same laws as before but for a more general flow.

The velocity field can now be eliminated from the Lagrangian density by simple substitution according to the Clebsch representation. This move could be dangerous; when inserting a solution of an Euler-Lagrange equation back into the Lagrangian density, the variation of the action with respect to a specific field vanishes automatically. Here, this field is the eliminated variable itself, so

[8] This row is labeled Euler's equation even though that equation does not follow immediately from setting this term to zero.

[9] The origin of this requirement is interesting. Seliger and Whitham attribute the idea to Chia-Chiao Lin, who introduced a vector field conserved along the flow as a way of guaranteeing that a Lagrangian description of the flow can exist. It turns out, however, that continuity of a scalar field is enough to achieve a Clebsch representation. They also note that "...Lin's device still remains somewhat mysterious from a strictly mathematical viewpoint, but the necessity for it seems to be firmly established as we proceed".

the other field equations remain intact. The set of equations in terms of v_i can then be recovered when needed by reinstating the solved equation by hand.

Eliminating the flow velocity in this manner reveals a simpler form for the Lagrangian density. After some manipulations using Leibniz's rule, the Lagrangian density is rewritten as (for convenience, the field κ has been replaced by $\rho\alpha = \kappa$, where α is a new scalar field)

$$\begin{aligned}&-\frac{1}{2}\rho\left(\partial_i\lambda + \alpha\partial_i\beta\right)\left(\partial_i\lambda + \alpha\partial_i\beta\right) - \frac{1}{8\pi}\partial_i\phi\partial_i\phi - \rho\phi - \rho\dot{\lambda} - \rho\alpha\dot{\beta}\\ &+\partial_t\left(\lambda\rho + \rho\alpha\beta\right) + \partial_i\left(\rho\lambda\left(\partial_i\lambda + \alpha\partial_i\beta\right) + \beta\rho\alpha\left(\partial_i\lambda + \alpha\partial_i\beta\right)\right).\end{aligned} \tag{1.63}$$

The last two terms will contribute surface terms to the action, which both vanish under the earlier stated boundary conditions. Hence, the Lagrangian density

$$\mathscr{L} = -\rho\dot{\lambda} - \rho\alpha\dot{\beta} - \frac{1}{2}\rho\left(\partial_i\lambda + \alpha\partial_i\beta\right)\left(\partial_i\lambda + \alpha\partial_i\beta\right) - \rho\phi - \frac{1}{8\pi}\partial_i\phi\partial_i\phi \tag{1.64}$$

is equivalent to the one in equation (1.59) in the sense that the corresponding action is stationary under the same conditions.

To complete this part of the investigation, the variation of the action must be written out one last time. The action integral

$$S\left[\alpha, \beta, \lambda, \rho, \phi\right] = \int_{t_I}^{t_F} \mathrm{d}t \int_\Omega \mathrm{d}^3x\, \mathscr{L}, \tag{1.65}$$

with the Lagrangian density (1.64), is stationary when the variation (no row labels this time)

$$\begin{aligned}\delta S = \int_{t_I}^{t_F} \mathrm{d}t \int_\Omega \mathrm{d}^3x \Bigg(&- \partial_t\left(\rho\delta\lambda\right) - \partial_t\left(\rho\alpha\delta\beta\right)\\ -\partial_i\left(\rho\partial_i\lambda\delta\lambda\right) - \partial_i\left(\rho\alpha^2\partial_i\beta\delta\beta\right) &- \partial_i\left(\rho\alpha\partial_i\lambda\delta\beta\right) - \partial_i\left(\rho\alpha\partial_i\beta\delta\lambda\right) - \frac{1}{4\pi}\partial_i\left(\partial_i\phi\delta\phi\right)\\ &-\delta\alpha\left(\rho\dot{\beta} + \rho\left(\partial_i\lambda + \alpha\partial_i\beta\right)\partial_i\beta\right)\\ &+\delta\beta\left(\partial_t\left(\rho\alpha\right) + \partial_i\left(\rho\alpha\left(\partial_i\lambda + \alpha\partial_i\beta\right)\right)\right)\\ &+\delta\lambda\left(\dot{\rho} + \partial_i\left(\rho\left(\partial_i\lambda + \alpha\partial_i\beta\right)\right)\right)\\ &-\delta\rho\left(\dot{\lambda} + \alpha\dot{\beta} + \phi + \frac{1}{2}\left(\partial_i\lambda + \alpha\partial_i\beta\right)\left(\partial_i\lambda + \alpha\partial_i\beta\right)\right)\\ &+\delta\phi\left(\frac{1}{4\pi}\nabla^2\phi - \rho\right)\Bigg)\end{aligned} \tag{1.66}$$

vanishes. From this, the set of field equations (with $D_t = \partial_t + (\partial_j\lambda + \alpha\partial_j\beta)\partial_j$)

$$\begin{cases} D_t\beta = 0 \qquad \text{(wherever} \quad \rho \neq 0),\\ D_t\alpha = 0 \qquad \text{(wherever} \quad \rho \neq 0),\\ \dot{\rho} + \partial_i\left(\rho\left(\partial_i\lambda + \alpha\partial_i\beta\right)\right) = 0,\\ \dot{\lambda} + \alpha\dot{\beta} + \phi + \frac{1}{2}\left(\partial_i\lambda + \alpha\partial_i\beta\right)\left(\partial_i\lambda + \alpha\partial_i\beta\right) = 0,\\ \nabla^2\phi = 4\pi\rho.\end{cases} \tag{1.67}$$

is derived. By reinstating the velocity field it is possible to get the nicer looking set

$$\begin{cases} \dot{\rho} + \partial_i \left(\rho v_i\right) = 0, & \text{Mass Continuity Equation} \\ \nabla^2 \phi = 4\pi\rho, & \text{Poisson's Equation} \\ D_t v_i = -\partial_i \phi. & \text{Euler's Equation} \end{cases} \tag{1.68}$$

1.4.3 Noether's theorem; energy and energy flux density

In the last section, a lot of work went into deriving three already familiar equations. In this section the effort invested in making ρ dynamical pays off, since the action integral lets us proceed with a Noetherian analysis of the implied conserved currents. The question at hand is about energy, so time translation symmetry is studied here. Other symmetries could be investigated following similar steps.

To find conserved currents in general, study an action integral like the one in the previous section but the integral taken over a region in space ω and a time interval $[t_i, t_f]$ such that they are both within the bounds of Ω and $[t_I, t_F]$. For distinction, this integral will be called S_ω and is given by

$$S_\omega \left[\alpha, \beta, \lambda, \rho, \phi\right] = \int_{t_i}^{t_f} \mathrm{d}t \int_\omega \mathrm{d}^3 x \, \mathscr{L}, \tag{1.69}$$

where the Lagrangian density is that of equation (1.64). An arbitrary variation of this functional with respect to the fields is of the same form as what was found for S, but now the surface terms should not be assumed to vanish. The resulting functional differential, evaluated for solutions of the field equations is, then,

$$\begin{split} \delta S_\omega = \int_{t_i}^{t_f} \mathrm{d}t \int_\omega \mathrm{d}^3 x \Big(- \partial_t \left(\rho \delta\lambda\right) - \partial_t \left(\rho\alpha\delta\beta\right) \\ -\partial_i \left(\rho \partial_i \lambda \delta\lambda\right) - \partial_i \left(\rho\alpha^2 \partial_i \beta \delta\beta\right) - \partial_i \left(\rho\alpha\partial_i\lambda\delta\beta\right) - \partial_i \left(\rho\alpha\partial_i\beta\delta\lambda\right) - \frac{1}{4\pi}\partial_i \left(\partial_i\phi\delta\phi\right) \Big). \end{split} \tag{1.70}$$

Since the Euler-Lagrange equations are taken to be satisfied everywhere and the integration region is arbitrary, the above equation should hold true throughout the gravitational system.

To find a conservation law for energy specifically, the variations in the fields should be those corresponding to a time translation. Such a change in a field ψ,

$$\psi(x_i, t + \epsilon) - \psi(x_i, t) \tag{1.71}$$

(where ϵ is a small constant) is, to first order,

$$\delta\psi = \epsilon\dot{\psi}. \tag{1.72}$$

At the same time, the Lagrangian density changes according to

$$\mathscr{L}\left(\psi(x_i, t+\epsilon), \dot{\psi}(x_i, t+\epsilon), \partial_i\psi(x_i, t+\epsilon)\right) - \mathscr{L}\left(\psi(x_i, t), \dot{\psi}(x_i, t), \partial_i\psi(x_i, t)\right) \tag{1.73}$$

(with ψ standing in for all the fields on which $\mathscr{L}$ depends), which similarly gives

$$\delta\mathscr{L} = \epsilon\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{L}. \tag{1.74}$$

Suppose now that all the fields are subject to such a change. Then there is, on the one hand, the change

$$\begin{aligned}\delta S_\omega = -\epsilon\int_{t_i}^{t_f}\mathrm{d}t\int_\omega \mathrm{d}^3x\Bigg(&\partial_t\left(\rho\dot\lambda+\rho\alpha\dot\beta\right)\\ +\partial_i\left(\rho\left(\dot\lambda+\alpha\dot\beta\right)(\partial_i\lambda+\alpha\partial_i\beta)+\frac{1}{4\pi}\dot\phi\partial_i\phi\right)&\Bigg)\end{aligned} \tag{1.75}$$

in the action from equation (1.70). On the other hand, the same change should be given by

$$\delta S_\omega = \int_{t_i}^{t_f}\mathrm{d}t\int_\omega \mathrm{d}^3x\,\delta\mathscr{L} = \epsilon\int_{t_i}^{t_f}\mathrm{d}t\int_\omega \mathrm{d}^3x\,\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{L}. \tag{1.76}$$

The implied equality results in the conservation law

$$-\partial_t\left(\rho\dot\lambda+\rho\alpha\dot\beta\right)-\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{L}-\partial_i\left(\rho\left(\dot\lambda+\alpha\dot\beta\right)(\partial_i\lambda+\alpha\partial_i\beta)+\frac{1}{4\pi}\dot\phi\partial_i\phi\right)=0, \tag{1.77}$$

or

$$\begin{aligned}&\partial_t\left(\frac{1}{2}\rho\left(\partial_i\lambda+\alpha\partial_i\beta\right)\left(\partial_i\lambda+\alpha\partial_i\beta\right)+\rho\phi+\frac{1}{8\pi}\partial_i\phi\partial_i\phi\right)\\ &-\partial_i\left(\rho\left(\dot\lambda+\alpha\dot\beta\right)(\partial_i\lambda+\alpha\partial_i\beta)+\frac{1}{4\pi}\dot\phi\partial_i\phi\right)=0\end{aligned} \tag{1.78}$$

with the time derivative of the Lagrangian density written out in terms of the fields. This result conforms to the expected form

$$\dot\varepsilon+\partial_i S_i = 0. \tag{1.79}$$

As a final step, reintroduce the velocity field as

$$v_i = \partial_i\lambda+\alpha\partial_i\beta, \tag{1.80}$$

and use the field equation for $\delta S/\delta\rho$, which is rewritten as

$$\dot\lambda+\alpha\dot\beta = -\phi-\frac{1}{2}v_iv_i, \tag{1.81}$$

to eliminate the Clebsch potentials from the energy expressions. This gives the result

$$\partial_t\left(\tau+\rho\phi+\frac{1}{8\pi}\partial_i\phi\partial_i\phi\right)+\partial_i\left(\left(\tau+\rho\phi\right)v_i-\frac{1}{4\pi}\dot\phi\partial_i\phi\right)=0 \tag{1.82}$$

(with $\tau = \rho v_iv_i/2$). Now, this is starting to look familiar.

The canonical form of energy density in Newtonian gravitation, then, is

$$\varepsilon = \tau+\rho\phi+\partial_i\phi\partial_i\phi \tag{1.83}$$

and the energy flux density is

$$S_i = (\tau + \rho\phi)\, v_i - \frac{1}{4\pi}\dot{\phi}\partial_i\phi. \tag{1.84}$$

Comparing to equation (1.22), it is clear that this result demands that $N = 2$. It is encouraging that this is the same result as was guessed at earlier. It should be noted, however that the freedom to transform this law into other forms is still in place.

1.5 Concluding remarks

In this chapter, the general ideas of a Newtonian field theory have been introduced. An arbitrariness regarding how to view local energy in such a theory is apparent, echoing the familiar situation in electrostatics. Two ideas for fixing the form of gravitational energy density were explored. One argument studies the non-relativistic limit of an energy expression in general relativity. The other argument is self-contained within Newtonian gravitation and relies on Noether's theorem to find an expression for energy density. As a bonus the non-trivial task of finding a variational principle for Newtonian field theory had to be undertaken, which turned out to be educational in its own right. We were pleasantly surprised to find that these two approaches support the same case; $N = 2$ in equation (1.22).

When it comes to the variational approach, it is clear which form for energy density is the canonical one, but this does not automatically discredit the alternatives. In electrodynamics, for example, the canonical energy-momentum tensor is actually not generally seen as correct, as it is neither symmetric nor gauge-invariant.

The correct form for electrodynamics is found by comparison to general relativity. Ideally, there would be some way of using Einstein's more complete theory to decide what the Newtonian energy expressions should be as well. Regarding the argument presented in this chapter, it is not clear that the energy density of Newtonian field theory necessarily corresponds to the non-relativistic limit of the relativistic matter energy density, as was guessed. In the introduction to this thesis we suggested that Newton-Cartan theory could be helpful in this matter, but it remains to be seen if this is the case.

Chapter 2

Energy Transfer by Gravitational Induction

While not as widely discussed as gravitational waves today, inductive transfer via gravitation is not a very exotic phenomenon. As Bondi points out [26, p. 431], energy is transmitted inductively from the earth to the earth-moon gravitational system via tidal friction. As this friction causes the earths rotation to slow down, the orbital altitude of the moon also increases. Some of the earths rotational energy has been turned into gravitational energy. It is possible to model this interaction in Newtonian gravitation, but (as was seen in chapter 1) such gravitational energy is not unambiguously localized. It is therefore rather unclear where in space this energy has traveled to.

Bondi and McCrea were able to construct a purer example of induction, where the system returns to its initial state after some energy has been transmitted. In their 1960 paper [1], the tidal forces between two gravitating bodies are used to move energy from the internal workings (external to the system) of one body to the other. Bondi has called this *the experiment of Tweedledum and Tweedledee* [26, p. 431]. In this chapter, their thought experiment is examined in detail.

As a follow-up to their study, it would be interesting to evaluate and visualize the energy density flux vectors of chapter 1. This task is almost impossibly tedious by hand, so it is not so hard to see why no one did it back in the late fifties. Computer technology has advanced significantly since then, however. At the end of this chapter, the findings of Bondi and McCrea are applied towards constructing a computer model of the energy-exchanging gravitating bodies. The energy flux density vectors of chapter 1 can then be evaluated by computer algebra.

First, a sidenote on notation. In the following sections, a more traditional vector notation (and the ∇ symbol) will be used rather than the conventions of the last chapter. This reveals the chronology of this project, as most of the work in the current chapter was done before we realized how useful the field theoretical approach is in these matters. Attempts at switching to a consistent notation proved more detrimental than helpful towards readability, so it has been left as is.

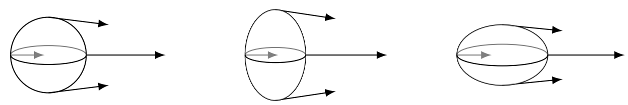

Figure 2.1: Illustration of the force field (arrows in the figure) on the surface of a body, caused by a spherical body somewhere to the right of it. The gravitational field exerts a flattening tidal force on a spherical body (a). Work needs to be done against the gravitational field in order to make the body more prolate (b), while work will be received for allowing it to become more oblate (c).

2.1 The experiment of Tweedledum and Tweedledee

Imagine that Tweedledum and Tweedledee are two mutually gravitating bodies, initially of spherical shape. Choose a plane that contains the two centers of mass and call this the orbital plane. The normal to this plane will be called the axial direction (the system will possess a mirror symmetry in this plane, so up or down does not matter). Imagine further that they both have the mechanical faculties to not only maintain their shape, but also deform themselves into being more oblate or prolate spheroids. That is, they can be prolonged or shortened in the axial direction, while at the same time reducing or enlarging their intersection with the orbital plane. It is clear from the axial symmetry of the setup that, even when deformed, the total attractive force between the bodies acts on the centers of each spheroid.

Such deformations cannot, however, be done for free. Tidal forces between the bodies want to flatten them, and resist the prolonging motion (study figure 2.1 to see this). As a consequence, becoming more prolate costs energy while energy can be extracted from the system by going more oblate. Because the tidal forces are stronger when Tweedledum and Tweedledee are close together, it is also clear that they gain or expend more work to deform if they are close than if they are far apart.

Were the two bodies always perfectly spherical, it is known that the orbits would form two similar ellipses with a common focus. To not spiral inward or outward, the pair agrees to only deform their shape in such a way that these Kepler orbits are preserved. That is, they must work in tandem so that the total attractive force is unchanged from the usual inverse square law. They both intuitively know exactly how to do this, but for phycisists it takes some work (sections 2.2, 2.3, and 2.4) just to accept that it is possible.

More energy is to be gained at the periapsis than at any other point in the cycle. Conversely, this is also when there is the most to be lost. Tweedledum, the smarter of the two, understands this, but Tweedledee does not. In their agreement, Tweedledum has snuck in a clause stipulating that he should be allowed to flatten at the closest approach. Tweedledee, the poor fool, must then do a great deal of work against the tidal force to honor their agreement, as he must make himself more prolate to maintain the magnitude of the force. When he gets to go oblate, he is guaranteed to receive less than Tweedledum did! Tweedledee has been tricked into expending work to preserve the orbits, while Tweedledum siphons it off and stores it in his belly. It is clear that the energy gained by Tweedledum must come from Tweedledee's internals and not from their mutual potential energy, since the system is assumed to return to the initial condition after a cycle of the orbits.

The above story raises some questions about its validity. Mainly, whether it is really true that the orbits can be preserved, whether this entails the counter phase deformations claimed, and whether the amount of transmitted energy can be quantified in some meaningful way. The mentioned paper by Bondi and McCrea does include a detailed quantifiable model as an appendix, but they skip over a lot of calculations and apply some quite intuitive arguments. While convincing, their paper is not so accessible to a wider audience. In the following sections (2.2-2.6), we reexamine their calculations and arguments and fill in some of the blanks. Some comments in the original paper also get interpreted to the best of our ability.

2.2 Gravitational potential of multipoles

Tweedledum and Tweedledee, who will be called receiver and transmitter for convenience, are each described by a mass density of the form (in spherical coordinates (r, θ, ϕ) centered on the body)

$$\rho = \begin{cases} \rho_0 \left(1 + \epsilon(t) \left(\frac{r}{r_0}\right)^2 P_2(\cos\theta)\right), & r < r_0, \\ 0, & r \geq r_0, \end{cases} \tag{2.1}$$

where P_2 is the Legendre polynomial

$$P_2(\cos\theta) = \frac{1}{2}\left(3\cos^2\theta - 1\right). \tag{2.2}$$

The constants ρ_0 and r_0, and the time dependent parameter ϵ are allowed to differ between transmitter and receiver. A positive ϵ captures the notion of being "more prolate" as it was called previously. That is, mixing in more of the second term redistributes mass from the equatorial plane to the poles. Similarly, a negative ϵ moves mass towards the equatorial plane. Another way of saying the same thing is that the distributions have each been given a variable spherical quadrupole moment. The monopole moment is constant and is of course just the total mass of the body, as is seen by solving the integral

$$\rho_0 \int \mathrm{d}^3 r + \frac{\rho_0 \epsilon}{r_0^2} \int_0^{r_0} \mathrm{d}r \int_0^{\pi} \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi \, r^4 \sin\theta P_2(\cos\theta). \tag{2.3}$$

The second term here is zero because it contains the integral of the product of two orthogonal polynomials,

$$\int_0^{\pi} \mathrm{d}\theta \, \sin\theta P_2(\cos\theta) = \int_{-1}^{1} \mathrm{d}x \, P_0(x) P_2(x) = 0. \tag{2.4}$$

And so, the monopole moment is just

$$M = \frac{4}{3}\pi r_0^3 \rho_0, \tag{2.5}$$

which does not change with ϵ (and therefore not with time).

This mass distribution is a more or less realistic model of a body that can become more prolate or more oblate at will. It suffers from the defect that it does not go smoothly towards zero at its surface; it is discontinuous at the distance r_0 from the center. This issue will crop up later when doing vector calculus on these objects, but that can be fixed when it needs to be. The expression above will suffice for the time being.

To proceed, one wants to find the potential outside such a mass distribution. This potential can be written as two terms;

$$V = MU_{\mathrm{M}} + QU_{\mathrm{Q}}, \tag{2.6}$$

where the quantities U_{M} and U_{Q}, then, are the "potential per moment" for the monopole and the quadrupole, respectively. The monopole potential,

$$U_{\mathrm{M}} = -\frac{1}{r}, \tag{2.7}$$

is very well known, but finding the quadrupole potential is educational enough to warrant a brief digression. The potential from the quadrupole is given by

$$-\epsilon\frac{\rho_0}{r_0^2}\int\limits_{r'<r_0} \mathrm{d}^3r'\,\frac{r'^2 P_2(\cos\theta')}{|\mathbf{r}-\mathbf{r}'|}. \tag{2.8}$$

Here, the integrand contains the generating function of the Legendre polynomials,

$$\frac{1}{|\mathbf{r}'-\mathbf{r}|} = \sum_{l=0}^{\infty}\frac{r'^l}{r^{l+1}}P_l(\cos\delta), \tag{2.9}$$

where δ is the angle between $\mathbf{r}$ and $\mathbf{r}'$. Now, the *addition theorem for spherical harmonics* states that this Legendre polynomial can be written as [12, pp. 110–111]

$$P_l(\cos\delta) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_{l,m}(\theta,\phi)Y^*_{l,m}(\theta',\phi'), \tag{2.10}$$

where the arguments of the spherical harmonics are the polar and azimuthal angles of $\mathbf{r}$ and $\mathbf{r}'$, respectively. Since the Legendre polynomial in the integral (2.8) can be rewritten as

$$P_2(\cos\theta') = 2\sqrt{\frac{5}{\pi}}Y_{2,0}(\theta',\phi'), \tag{2.11}$$

the addition theorem implies the unwieldy expression

$$QU_{\mathrm{Q}} = -\epsilon\frac{8\sqrt{5\pi}\rho_0}{r_0^2}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{Y_{l,m}(\theta,\phi)}{r^{l+1}\,(2l+1)}\int_0^{r_0}\mathrm{d}r'\,r'^{l+4}\int\mathrm{d}\Omega'\,Y_{2,0}(\theta',\phi')Y^*_{l,m}(\theta',\phi'), \tag{2.12}$$

for the quadrupole potential. Luckily, the spherical harmonics are orthonormal, in the sense of

$$\int\mathrm{d}\Omega'\,Y_{2,0}(\theta',\phi')Y^*_{l,m}(\theta',\phi') = \delta_{l,2}\delta_{m,0}, \tag{2.13}$$

so it all simplifies to

$$QU_{\mathrm{Q}} = -\epsilon\frac{4\pi}{35}r_0^5\rho_0\frac{P_2(\cos\theta)}{r^3}, \tag{2.14}$$

and that is how you find the potential from a multipole. As a bonus, the quadrupole moment

$$Q = \epsilon\frac{4\pi}{35}r_0^5\rho_0 \tag{2.15}$$

can be read off, and it varies in time with ϵ as was promised.

In terms of multipole moments, the two bodies are described by

$$\rho_{\mathrm{T}} = \frac{3}{4\pi}\frac{M_{\mathrm{T}}}{r_{\mathrm{T}}^3} + \frac{35}{4\pi}\frac{Q_{\mathrm{T}}}{r_{\mathrm{T}}^7} r^2 P_2(\cos\theta), \tag{2.16}$$

and

$$\rho_{\mathrm{R}} = \frac{3}{4\pi}\frac{M_{\mathrm{R}}}{r_{\mathrm{R}}^3} + \frac{35}{4\pi}\frac{Q_{\mathrm{R}}}{r_{\mathrm{R}}^7} r^2 P_2(\cos\theta), \tag{2.17}$$

where it is understood that the spherical coordinates are centered on each body. Bondi and McCrea used equation (2.1), but here the expressions above in terms of multipole moments will be used instead, as it is more convenient in some of the steps to follow.

2.3 Potential energy of multipoles

Several interesting quantities can be had as partial derivatives of the potential energy, defined as

$$W = \int \mathrm{d}^3 r\, \rho_{\mathrm{T}} V_{\mathrm{R}}, \tag{2.18}$$

where the mass density from equation (2.16) (the transmitter) is placed in the potential

$$V_{\mathrm{R}} = M_{\mathrm{R}} U_{\mathrm{M}}(\mathbf{r} - \mathbf{D}) + Q_{\mathrm{R}} U_{\mathrm{Q}}(\mathbf{r} - \mathbf{D}). \tag{2.19}$$

This potential is that generated by the receiver, translated by a separating vector $\mathbf{D}$. This particular choice may seem a little artificial, given that it looks like the transmitter is fixed to the origin. Such a choice of inertial frame can, however, be found at any instant. Because of the translation, one finds

$$\frac{\partial V_{\mathrm{R}}}{\partial D} = \nabla V_{\mathrm{R}} \cdot \frac{\partial}{\partial D}(\mathbf{r} - \mathbf{D}) = -\nabla V_{\mathrm{R}} \cdot \widehat{\mathbf{D}} \tag{2.20}$$

($\widehat{\mathbf{D}}$ is the unit vector along $\mathbf{D}$). Using this in the derivative

$$\frac{\partial W}{\partial D} = \int \mathrm{d}^3 r\, \rho_{\mathrm{T}} \frac{\partial V_{\mathrm{R}}}{\partial D} \tag{2.21}$$

gives

$$\frac{\partial W}{\partial D} = -\int \mathrm{d}^3 r\, \rho_{\mathrm{T}} \nabla V_{\mathrm{R}} \cdot \widehat{\mathbf{D}}. \tag{2.22}$$

So, the D-derivative of the potential energy is the magnitude of the force on the transmitter in the direction towards the other body. Given the assumed symmetries, this is also the full magnitude of the attractive force. As mentioned, it will turn out that more useful quantities can be had as such derivatives.

Finding this energy, then, is a good way to proceed. It may be helpful to think of the system as consisting of four objects; two monopoles and two quadrupoles, where the potential energy of overlapping multipoles is not counted towards the potential energy. Thus, the potential energy may be decomposed as

$$W = W_{\mathrm{MM}} + W_{\mathrm{MQ}} + W_{\mathrm{QM}} + W_{\mathrm{QQ}}, \tag{2.23}$$

where W_{MM} is the interaction between the two monopoles, W_{MQ} and W_{QM} are the interactions between quadrupoles and monopoles, and W_{QQ} is the interaction between quadrupoles. The first three terms are easy to find. The monopole interaction is

$$W_{\mathrm{MM}} = -\frac{M_{\mathrm{T}} M_{\mathrm{R}}}{D}. \tag{2.24}$$

To find the interaction between quadrupole and monopole, simply place the mass of the monopole in the potential from the quadrupole. This is done by evaluating the expression

$$W_{\mathrm{MQ}} = M_{\mathrm{T}} Q_{\mathrm{R}} U_{\mathrm{Q}} \Big|_{r=D, \theta=\pi/2}, \tag{2.25}$$

and in this way the sum of monopole-quadrupole interactions is found to be

$$W_{\mathrm{MQ}} + W_{\mathrm{QM}} = \frac{1}{2D^3} \left(M_{\mathrm{T}} Q_{\mathrm{R}} + M_{\mathrm{R}} Q_{\mathrm{T}} \right). \tag{2.26}$$

Writing out the quadrupole-quadrupole interaction gives

$$W_{\mathrm{QQ}} = -7 \frac{Q_{\mathrm{T}} Q_{\mathrm{R}}}{r_{\mathrm{T}}^7} \int\limits_{r<r_{\mathrm{T}}} \mathrm{d}^3 r \, \frac{r^2}{|\mathbf{r} - \mathbf{D}|^3} Y_{2,0}(\mathbf{r}) Y_{2,0}(\mathbf{r} - \mathbf{D}), \tag{2.27}$$

where the arguments of the spherical harmonics are to be interpreted as the polar and azimuthal angles of the vectors written out. But alas, there is a complication. How do you handle the translation in one of the arguments? Since this problem is also of interest to those who study inter-molecular forces, the solution to the equivalent electrostatic problem can be found in the relevant literature (for example in [21] in terms of discrete charges or in [23, pp. 45–47] in terms of multipole moments). It is worthwhile to see how the the answer comes about from the transformation properties of spherical harmonics, however.

Translated spherical harmonics can be expanded in a useful series. For the irregular solid harmonic needed here, there is the formula [8, p. 65]

$$\frac{1}{|\mathbf{r} - \mathbf{D}|^{l+1}} Y_{l,m}(\mathbf{r} - \mathbf{D}) = (-1)^m \sum_{l'=0}^{\infty} \sum_{m'=-l'}^{l'} C_{l,l'}^{m,m'} \frac{r^{l'}}{D^{l+l'+1}} Y_{l+l',m+m'}(\mathbf{D}) Y_{l',m'}^*(\mathbf{r}) \tag{2.28}$$

(assuming that D is larger than r). The coefficients $C_{l,l'}^{m,m'}$ are given by[1]

$$C_{l,l'}^{m,m'} = \sqrt{\frac{4\pi(2l+1)}{(2l'+1)(2l'+2l+1)}} \sqrt{\frac{(l+m+l'+m')!(l-m+l'-m')!}{(l+m)!(l-m)!(l'+m')!(l'-m')!}}. \tag{2.29}$$

Applying this to equation (2.27), with $l = 2$ and $m = 0$, there is once again a product of orthonormal functions ($Y_{2,0}(\mathbf{r})$ and $Y_{l',m'}^*(\mathbf{r})$), so every term except one will vanish when doing the angular integrals. The relevant term to keep is

$$W_{\mathrm{QQ}} = -7 \frac{Q_{\mathrm{T}} Q_{\mathrm{R}}}{r_{\mathrm{T}}^7} \int\limits_{r<r_{\mathrm{T}}} \mathrm{d}^3 r \left(\ldots + C_{2,2}^{0,0} \frac{r^4}{D^5} Y_{4,0}(\mathbf{D}) Y_{2,0}(\mathbf{r}) Y_{2,0}^*(\mathbf{r}) + \ldots \right). \tag{2.30}$$

[1] There is a misprint in the source. The coefficient in the second equation on page 65 of [8] should be as stated here.

After doing the integrals, only the expression

$$W_{\mathrm{QQ}} = -\frac{Q_{\mathrm{T}} Q_{\mathrm{R}}}{D^5} C^{0,0}_{2,2} Y_{4,0}(\mathbf{D}) \tag{2.31}$$

remains. The axially symmetric spherical harmonic $Y_{4,0}$ can be traded for a Legendre polynomial ($C^{0,0}_{2,2} Y_{4,0} = 6P_4$), and the polar angle of the separation vector is always $\pi/2$, so this can further be simplified to

$$W_{\mathrm{QQ}} = -6\frac{Q_{\mathrm{T}} Q_{\mathrm{R}}}{D^5} P_4(\cos \pi/2). \tag{2.32}$$

The result for quadrupole-on-quadrupole interaction is, then,

$$W_{\mathrm{QQ}} = -\frac{9}{4}\frac{Q_{\mathrm{T}} Q_{\mathrm{R}}}{D^5}. \tag{2.33}$$

The result for the whole potential energy is

$$W = -\frac{M_{\mathrm{T}} M_{\mathrm{R}}}{D} + \frac{1}{2}\frac{M_{\mathrm{T}} Q_{\mathrm{R}} + M_{\mathrm{R}} Q_{\mathrm{T}}}{D^3} - \frac{9}{4}\frac{Q_{\mathrm{T}} Q_{\mathrm{R}}}{D^5}. \tag{2.34}$$

2.4 Kepler orbits

In the introduction to this chapter it was claimed that Tweedledum and Tweedledee can agree to preserve their Kepler orbits, even while deforming into non-spherical shapes. It is now time to show that this is indeed possible. From the results in the previous section, the magnitude of the attractive force is

$$F = \frac{\partial W}{\partial D} = \frac{M_{\mathrm{T}} M_{\mathrm{R}}}{D^2} - \frac{3}{2}\frac{M_{\mathrm{T}} Q_{\mathrm{R}} + M_{\mathrm{R}} Q_{\mathrm{T}}}{D^4} + \frac{45}{4}\frac{Q_{\mathrm{T}} Q_{\mathrm{R}}}{D^6}. \tag{2.35}$$

Now, if the quadrupoles satisfy the condition

$$Q_{\mathrm{T}} = -\frac{2M_{\mathrm{T}} Q_{\mathrm{R}}}{2M_{\mathrm{R}} - 15Q_{\mathrm{R}}/D^2}, \tag{2.36}$$

the two last terms cancel at all times. This (together with the symmetries of the setup) makes the total force on the receiver

$$\mathbf{F} = -\frac{M_{\mathrm{T}} M_{\mathrm{R}}}{D^3}\mathbf{D}, \tag{2.37}$$

where $\mathbf{D}$ is the separation between the bodies (pointing from transmitter to receiver). This essentially solves the problem of the orbits. They can be the known ellipses even with varying quadrupoles, as long as the condition above is upheld.

When introducing this thought experiment, a silent assumption was made that the two quadrupoles should vary in opposite phase. According to the condition above, however, this is only half true. The counter-phase relationship is modulated by the factor

$$\frac{2M_{\mathrm{T}}}{2M_{\mathrm{R}} - 15Q_{\mathrm{R}}/D^2}. \tag{2.38}$$

By choosing the two masses to be similar and D to be large, this factor can be made arbitrarily close to one. It remains to be seen that Q_{R} can be specified as a function of time such that energy

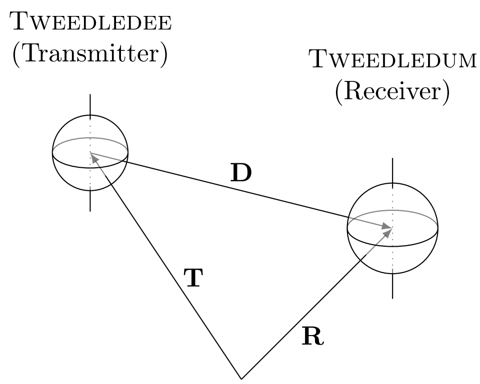

Figure 2.2: The two bodies as seen from an arbitrary frame of reference.

is transmitted, but that is content for the next section. The rest of this section contains some useful details about the classical two-body problem.

The solution of the two-body problem follows the standard procedure of turning it into an equivalent one-body problem concerning only the separation vector between the two bodies. As a reminder, a brief explanation is presented below (see [9, ch.3] for a detailed treatment).

The situation is illustrated in figure 2.2, where the transmitter is found at the position $\mathbf{T}$, and the receiver at $\mathbf{R}$. The motion of the separation vector is given by Newton's second law,

$$\ddot{\mathbf{D}} = \ddot{\mathbf{R}} - \ddot{\mathbf{T}} = \frac{1}{M_{\mathrm{R}}}\mathbf{F}_{\mathrm{R}} - \frac{1}{M_{\mathrm{T}}}\mathbf{F}_{\mathrm{T}}, \tag{2.39}$$

where $\mathbf{F}_{\mathrm{R}}$ is the total force on the receiver, and $\mathbf{F}_{\mathrm{T}}$ is the total force on the transmitter. By Newton's third law, these two forces should be equal in magnitude and of opposite direction; $\mathbf{F}_T = -\mathbf{F}_R$. The acceleration of the separation vector is therefore

$$\ddot{\mathbf{D}} = \frac{1}{m}\mathbf{F}_{\mathrm{R}}, \tag{2.40}$$

in terms of the reduced mass

$$m = \frac{M_{\mathrm{T}}M_{\mathrm{R}}}{M_{\mathrm{T}} + M_{\mathrm{R}}}. \tag{2.41}$$

This motion of the separation is the equivalent one-body problem.

If the quadrupoles are subject to the constraint (2.36), the force obeys the usual inverse square law for a point mass. In that case, the initial conditions can be chosen such that $\mathbf{D}$ sweeps out an ellipse [9, p. 94], which one can write as

$$D = D_{\mathrm{min}}\frac{1+e}{1+e\cos\gamma}, \quad D_{\mathrm{min}} > 0, \quad 0 < e < 1 \tag{2.42}$$

(the parameter e is the eccentricity of the orbit). A convenient choice of inertial frame is to place the center of mass at the origin, have two of the axes span the equatorial plane, and one of those two axes aligned with the separation vector $\mathbf{D}$ when $\gamma = 0$. With this choice, illustrated in figure 2.3, the position of the transmitter can be written as

$$\mathbf{T} = -\frac{m}{M_{\mathrm{T}}}\mathbf{D} \tag{2.43}$$

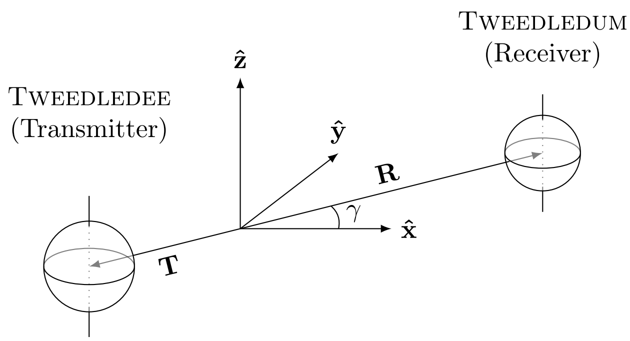

Figure 2.3: A convenient choice of frame. At the periapsis, $\gamma = 0$, and the separation between the bodies is parallel to $\hat{\mathbf{x}}$.

and that of the receiver as

$$\mathbf{R} = \frac{m}{M_\mathrm{R}}\mathbf{D}. \tag{2.44}$$

There are two globally conserved quantities in the one-body problem. The central force does not exert any torque, so there is a conserved angular momentum,

$$l = \left|\mathbf{D} \times \dot{\mathbf{D}}\right| = \left|\mathbf{D} \times \left(\dot{D}\widehat{\mathbf{D}} + D\dot{\gamma}\widehat{\boldsymbol{\gamma}}\right)\right| = D^2\dot{\gamma}. \tag{2.45}$$

This is useful for evaluating the time derivative $\dot{\gamma}$. Also, there is a conserved energy

$$E_1 = \frac{1}{2}m\dot{D}^2 + \frac{1}{2}\frac{ml^2}{D^2} - \frac{M_\mathrm{T}M_\mathrm{R}}{D}. \tag{2.46}$$

Note that this is not what would usually be called the total energy of the system, which is why it has been given the name E_1. The potential energy here is $-M_\mathrm{T}M_\mathrm{R}/D$, which is not the same as the potential energy W of the system itself. That is, using the condition (2.36) to eliminate one of the quadrupoles from W does not give the result $-M_\mathrm{T}M_\mathrm{R}/D$.

2.5 Finding the transmitted energy

For a potential depending on time via some parameters, there is a general sense in which change in the parameter is doing work on the system. Assume a simple one-dimensional mechanical system, with the equation of motion

$$\ddot{x} + \frac{\partial}{\partial x}V(x, \epsilon(t)) = 0. \tag{2.47}$$

Next, define a system energy

$$E = \frac{1}{2}\dot{x}^2 + V, \tag{2.48}$$

and compute its time derivative

$$\dot{E} = \dot{x}\left(\ddot{x} + \frac{\partial V}{\partial x}\right) + \frac{\partial V}{\partial \epsilon}\dot{\epsilon}. \tag{2.49}$$

The motion of x is such that the expression in parentheses is zero, but the energy is still not necessarily constant, because the parameter ϵ can exert mechanical power on the system. The work done during one cycle of a periodic system is therefore

$$\int_0^T \frac{\partial V}{\partial \epsilon} \dot{\epsilon} \, \mathrm{d}t, \tag{2.50}$$

or

$$\oint \frac{\partial V}{\partial \epsilon} \, \mathrm{d}\epsilon. \tag{2.51}$$

Seeing $\mathrm{d}\epsilon$ as a sort of displacement, then $\partial V / \partial \epsilon$ is (in the words of Bondi and McCrea) a "generalized force". The contour symbol is here used to remind the reader that integration is performed along a specific closed curve in the (x, ϵ)-space.

In the case of Tweedledum and Tweedledee, there are corresponding generalized forces exerted by the varying quadrupole moments. The work done on the system by deforming the receiver should therefore be

$$\oint \frac{\partial W}{\partial Q_{\mathrm{R}}} \, \mathrm{d}Q_{\mathrm{R}}.$$

Bondi and McCrea use this observation to find an integral expression of the transmitted energy. First, they compute the partial derivative

$$\frac{\partial W}{\partial Q_{\mathrm{R}}} = \frac{1}{2} \frac{1}{D^3} \left(M_{\mathrm{T}} - \frac{9}{2} \frac{Q_{\mathrm{T}}}{D^2} \right), \tag{2.52}$$

then use the requirement (2.36) between the quadrupoles to eliminate Q_{T},[2] and conclude that all the work done on the receiver during one cycle is

$$-\frac{1}{2} M_{\mathrm{T}} \oint \frac{2M_{\mathrm{R}}D^2 - 6Q_{\mathrm{R}}}{2M_{\mathrm{R}}D^2 - 15Q_{\mathrm{R}}} \frac{\mathrm{d}Q_{\mathrm{R}}}{D^3}. \tag{2.53}$$

The original authors give the above result with the opposite sign. It is possible that this is a typographical error, but it is also possible that they mean the work done on the receiver by an external force, as opposed to the work done by the system (which is what is intended here).

This shortcut seems intuitively correct, but the idea takes a little getting used to. To convince the reader, some complementary materials are presented in the two following subsections. First, it is shown that the idea of generalized forces, and the corresponding work done, works out the same in the present specific system as it did in the one-dimensional example above. Then, an alternative way of finding the same result is presented. It remains to be seen that the integral (2.53) actually represents a net transfer of energy, but this will be handled in section 2.6.

2.5.1 Generalized forces and work in the two-body problem

Given that the problem at hand can be reduced to a one-dimensional problem, it is reasonable to expect to find something similar to equations (2.47) to (2.51). Indeed, there is the one-dimensional equation of motion

$$m\ddot{\mathbf{D}} \cdot \widehat{\mathbf{D}} + \frac{\partial W}{\partial D} = 0, \tag{2.54}$$

[2]It is imperative that the condition is inserted after taking the partial derivative. Otherwise, the distinction between the two generalized forces is lost.

and the energy of the system may be defined as

$$E = \frac{1}{2}M_{\rm T}\left|\dot{\mathbf{T}}\right|^2 + \frac{1}{2}M_{\rm R}\left|\dot{\mathbf{R}}\right|^2 + W. \tag{2.55}$$

The sum of kinetic energies can be rewritten using the separation vector, as

$$\frac{1}{2}m\left|\dot{\mathbf{D}}\right|^2, \tag{2.56}$$

so the time derivative of the system energy is

$$\dot{E} = m\ddot{\mathbf{D}}\cdot\dot{\mathbf{D}} + \frac{\partial W}{\partial D}\dot{D} + \frac{\partial W}{\partial Q_{\rm T}}\dot{Q}_{\rm T} + \frac{\partial W}{\partial Q_{\rm R}}\dot{Q}_{\rm R}. \tag{2.57}$$

The first term simplifies according to

$$m\ddot{\mathbf{D}}\cdot\left(\dot{D}\widehat{\mathbf{D}} + D\dot{\gamma}\widehat{\boldsymbol{\gamma}}\right) = m\dot{D}\ddot{\mathbf{D}}\cdot\widehat{\mathbf{D}}, \tag{2.58}$$

because the force is always parallel to $\mathbf{D}$. The two first terms on the right hand side of equation (2.57) therefore always cancel. As expected, mechanical power is being supplied or absorbed by the deformation of the quadrupoles;

$$\dot{E} = \frac{\partial W}{\partial Q_{\rm T}}\dot{Q}_{\rm T} + \frac{\partial W}{\partial Q_{\rm R}}\dot{Q}_{\rm R}. \tag{2.59}$$

The change in system energy during a full period of the motion can be written as[3]

$$\Delta E = \oint \frac{\partial W}{\partial Q_{\rm T}}\,\mathrm{d}Q_{\rm T} + \oint \frac{\partial W}{\partial Q_{\rm R}}\,\mathrm{d}Q_{\rm R}. \tag{2.60}$$

Given that all the quantities on which E depends return to their initial values after one cycle of the orbits, one must conclude that the change ΔE is zero and that according to the result

$$\oint \frac{\partial W}{\partial Q_{\rm T}}\,\mathrm{d}Q_{\rm T} + \oint \frac{\partial W}{\partial Q_{\rm R}}\,\mathrm{d}Q_{\rm R} = 0,$$

all of the work found here cancels during one period. This is reassuring, as it means that Bondi and McCrea have not inadvertently constructed a perpetual motion machine of the first kind.

2.5.2 Alternative calculation

Bondi and McCrea concluded their article with the following puzzling words concerning equation (2.53)

> "The result can be checked by calculating the forces directly in the equivalent pole-quadrupole system without employing the energy W."

[3] For simplicity, it will be assumed that the quadrupoles are oscillating with the same period as the orbits. This seems like a sensible choice, and is in keeping with how the idea was originally put forward, but it is not strictly necessary.

The method presented here is a best guess as to what calculation they had in mind. It is a special case of the derivation of the Newtonian conservation law of chapter 1, which was inspired by Bondi's method of finding a Newtonian Poynting vector to begin with.

The calculation is easiest to do in a non-rotating frame accelerating with the receiver, rather than in a center-of-momentum frame. The acceleration of such a frame is

$$-\frac{M_{\mathrm{T}}}{D^3}\mathbf{D},$$

so the density of inertial force is

$$\rho_{\mathrm{R}}\frac{M_{\mathrm{T}}}{D^3}\mathbf{D}.$$

With V_{T} as the potential generated by the transmitting body, the apparent force density field at a point $\mathbf{r}$ can be written as the gradient of an "apparent potential" according to

$$-\rho_{\mathrm{R}}\nabla V_{\mathrm{T}}+\rho_{\mathrm{R}}\frac{M_{\mathrm{T}}}{D^3}\mathbf{D}=-\rho_{\mathrm{R}}\nabla\left(V_{\mathrm{T}}-\frac{M_{\mathrm{T}}}{D^3}\mathbf{D}\cdot\mathbf{r}\right). \tag{2.61}$$

This is the full force doing work as seen from this frame.

To get at the mechanical power, the velocity of the matter, $\mathbf{v}$ is introduced. As in chapter 1, it will be assumed that there is continuity of mass;

$$\dot{\rho}_{\mathrm{R}}+\nabla\cdot(\rho_{\mathrm{R}}\mathbf{v})=0. \tag{2.62}$$

This equation can be used to trade the unknown flow velocity for the known time derivative of the mass distribution.

The input of power to the receiver by forces originating in the transmitter is

$$-\int \mathrm{d}^3r\,\rho_{\mathrm{R}}\nabla\left(V_{\mathrm{T}}-\frac{M_{\mathrm{T}}}{D^3}\mathbf{D}\cdot\mathbf{r}\right)\cdot\mathbf{v}, \tag{2.63}$$

where the integral can be taken over all space or just over the volume of the receiver. Integration by parts gives

$$-\int_{\partial R}\mathrm{d}\mathbf{n}\cdot\mathbf{v}\rho_{\mathrm{R}}\left(V_{\mathrm{T}}-\frac{M_{\mathrm{T}}}{D^3}\mathbf{D}\cdot\mathbf{r}\right)+\int \mathrm{d}^3r\,\left(V_{\mathrm{T}}-\frac{M_{\mathrm{T}}}{D^3}\mathbf{D}\cdot\mathbf{r}\right)\nabla\cdot(\rho_{\mathrm{R}}\mathbf{v}) \tag{2.64}$$

after use of the divergence theorem (the surface ∂R can be taken to be any surface that encloses the receiver). If this application of the divergence theorem seems suspicious, hold on to that thought until the end of this section. Now, the mass density is zero outside the body, so the surface term vanishes. In the term that remains, continuity of mass lets the divergence be replaced by the time derivative of the mass distribution, which is

$$\dot{\rho}_{\mathrm{T}}=\frac{35}{4\pi}\frac{r^2}{r_{\mathrm{R}}^7}P_2(\cos\theta)\dot{Q}_{\mathrm{R}}. \tag{2.65}$$

In the term related to the inertial force,

$$-\frac{35}{4\pi}\frac{1}{r_{\mathrm{R}}^7}\dot{Q}_{\mathrm{R}}\frac{M_{\mathrm{T}}}{D^3}\mathbf{D}\cdot\int \mathrm{d}^3r\,r^2P_2(\cos\theta)\mathbf{r}, \tag{2.66}$$

the integral vanishes due to the odd parity of $\mathbf{r}$ (r^2 and $P_2(\cos\theta)$ are even). It has been discovered that in this case, the inertial force does no work at all. What remains is

$$\int \mathrm{d}^3r\, \dot{\rho}_\mathrm{R} V_\mathrm{T}. \tag{2.67}$$

To complete the calculation, the integral above is written out appropriately as

$$-7\frac{\dot{Q}_\mathrm{R}}{r_\mathrm{R}^7}\int \mathrm{d}^3r\, r^2 Y_{0,2}(\mathbf{r})\left(\sqrt{5}\frac{M_\mathrm{T}}{|\mathbf{r}+\mathbf{D}|}Y_{0,0}(\mathbf{r}+\mathbf{D}) + \frac{Q_\mathrm{T}}{|\mathbf{r}+\mathbf{D}|^3}Y_{2,0}(\mathbf{r}+\mathbf{D})\right). \tag{2.68}$$

Performing the translations and the angular integrals[4] gives

$$-\sqrt{4\pi}\frac{7}{r_\mathrm{R}^7}\frac{\dot{Q}_\mathrm{R}}{D^3}\int \mathrm{d}r\, r^4\left(\frac{1}{\sqrt{5}}M_\mathrm{T}Y_{2,0}(\mathbf{D}) + 2\frac{Q_\mathrm{T}}{D^2}Y_{4,0}(\mathbf{D})\right). \tag{2.69}$$

As with the potential energy, there are only axially symmetric harmonics. They are unaffected by the passage of time, and easily evaluated for $\theta = \pi/2$, which gives

$$\frac{1}{2}\frac{\dot{Q}_\mathrm{R}}{D^3}\left(M_\mathrm{T} - \frac{9}{2}\frac{Q_\mathrm{T}}{D^2}\right). \tag{2.70}$$

This is consistent with the result found with Bondi's and McCrea's quick method. Interestingly, requiring continuity of mass is necessary to get consistent results, but the result itself can be found without defining how matter is flowing.

Now this may all seem convincing, but the divergence theorem was applied rather dubiously. The mass distribution has been taken to have a discontinuity at the surface, so the continuity relation relation for mass does not necessarily make any sense there. That is, in the divergence

$$\nabla\cdot\left(\rho_\mathrm{R}\mathbf{v}\left(V_\mathrm{T} - \frac{M_\mathrm{T}}{D^3}\mathbf{D}\cdot\mathbf{r}\right)\right), \tag{2.71}$$

the expression inside the inner parentheses is sufficiently smooth, but the product $\rho_\mathrm{R}\mathbf{v}$ may drop to zero suddenly at the surface. This use of the divergence theorem cannot be allowed in general, as one could then make any volume integral over the divergence of a vector field vanish by simply setting the field to zero outside the integration domain.

The mass density can be made smooth by introducing *bump functions*; functions which go to zero in an infinitely differentiable manner. An example is a function of the form

$$\psi_n(r) = k_n e^{-\frac{r_0^2}{r_0^2 - r^2}}, \tag{2.72}$$

which goes smoothly towards zero as r approaches r_0 from below (k_n is a normalization factor). Each mass distribution is redefined according to the recipe

$$\rho = \frac{3}{4\pi}\frac{M}{r_0^3}\psi_1 + \frac{35}{4\pi}\frac{Q}{r_0^7}r^2 P_2(\cos\theta)\psi_2, \tag{2.73}$$

[4] These steps are similar to equations (2.28) to (2.31).

with the bump functions normalized such that the multipole moments are preserved;

$$\int_0^{r_0} \mathrm{d}r\ r^2\psi_1 = \frac{r_0^3}{3} \tag{2.74}$$

and

$$\int_0^{r_0} \mathrm{d}r\ r^6\psi_2 = \frac{r_0^7}{7}. \tag{2.75}$$

This, in turn, means that the potentials will also be preserved (see for example equation (2.12), where multiplying the integrand by ψ_2 would make no difference to the result), and also that the potential energy W and power input are unaffected (see equations (2.33) and (2.69) to be convinced). In this manner, vector calculus can be made to feel safe. The important take-away is that the earlier liberal use of the divergence theorem did not lead to a contradiction. Synge solves this problem with a different method [24, p. 377].

2.6 Quantifying the transmitted energy

The requirement (2.36) on the quadrupoles, meant to keep the bodies on elliptical orbits, allows each mass distribution to be rewritten in terms of the other quadrupole, for example

$$\rho_{\mathrm{T}} = \frac{3}{4\pi}\frac{M_{\mathrm{T}}}{r_{\mathrm{T}}^3} - \frac{35}{2\pi}\frac{1}{r_{\mathrm{R}}^7}\frac{M_{\mathrm{T}}Q_{\mathrm{R}}}{2M_{\mathrm{R}} - 15Q_{\mathrm{R}}/D^2}r^2P_2(\cos\theta). \tag{2.76}$$

Now, this presents an issue. It seems like the mass density can get infinitely large (or small) if the denominator in the second term is allowed to be zero. It is a sensible requirement, then, that the multipole moments and initial conditions must be chosen such that the inequalities

$$\begin{cases} \frac{15}{2}\frac{Q_{\mathrm{R}}}{M_{\mathrm{R}}} < D^2 \\ \frac{15}{2}\frac{Q_{\mathrm{T}}}{M_{\mathrm{T}}} < D^2 \end{cases} \tag{2.77}$$

both hold at all times. Another consequence of this requirement is that the integral

$$I = -\oint \frac{2M_{\mathrm{R}}D^2 - 6Q_{\mathrm{R}}}{2M_{\mathrm{R}}D^2 - 15Q_{\mathrm{R}}}\frac{\mathrm{d}Q_{\mathrm{R}}}{D^3}, \tag{2.78}$$

in the expression (2.53) for the energy transmitted during a cycle, simplifies somewhat in the sense that the integrand is always positive.

Bondi and McCrea write that the integral above (from now on called the work integral) “clearly does not vanish” if the receiver quadrupole varies “in quadrature” with the distance between the bodies. Some time has been spent trying to decipher this comment, and it turns out that “variation in quadrature” is the electrical engineers way of saying 90 degrees out of phase. Such a phase relationship, for a purely sinusoidal dependence upon γ in the quadrupole, is shown in figure 2.4. So, it is understood what variation in quadrature is, but is it so obvious that the integral does not vanish? The plot over the phases help with this. Assuming the separation distance dominates such that

$$\frac{2M_{\mathrm{R}}D^2 - 6Q_{\mathrm{R}}}{2M_{\mathrm{R}}D^2 - 15Q_{\mathrm{R}}} \approx 1 \tag{2.79}$$

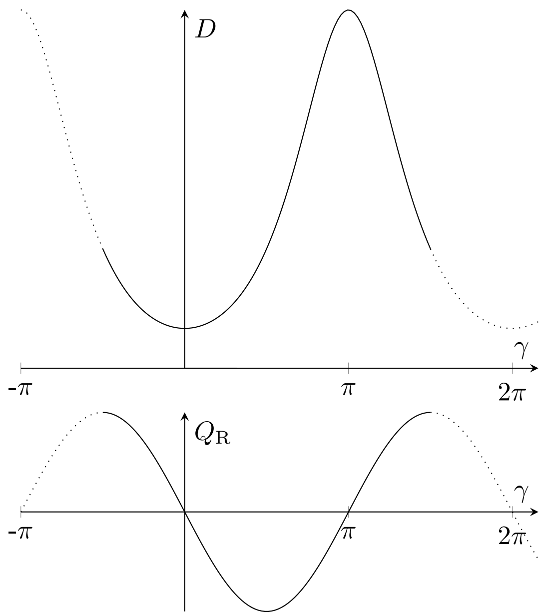

Figure 2.4: The illustrated phase relationship should be satisfied by the distance between the bodies (D) and the receiver quadrupole (Q_R). In this example, the separation $\mathbf{D}$ sweeps out an ellipse with the eccentricity one half.

(this ratio has already been assumed to always be positive), the work integral turns into just

$$I \approx -\oint \frac{\mathrm{d}Q_\mathrm{R}}{D^3}. \tag{2.80}$$

With the chosen phase relationship (study the figure), D is larger when Q_R is increasing (i.e. $\mathrm{d}Q_\mathrm{R}$ is positive) than it is when Q_R is decreasing ($\mathrm{d}Q_\mathrm{R}$ negative). Over a cycle, then, the work integral must be positive. This also entails that the work done by the system on the receiver during one cycle is positive. Thus, there is a net transfer of energy to it.

The approximation above is not necessary, as the work done is seemingly always positive (given elliptical orbits). This can be seen via a search of the parameter space as follows. Implement the suggested phase relationship by choosing the receiver quadrupole to be (as in figure 2.4)

$$Q_\mathrm{R} = -Q_\mathrm{R}^\mathrm{max} \sin\gamma, \tag{2.81}$$

where the coefficient is positive. Then replace the function D in the work integral by its solution as an ellipse, and rewrite the work integral into (the parameter γ can be used to compute the integral)

$$\frac{D_\mathrm{min}^3 (1+e)^3}{Q_\mathrm{R}^\mathrm{max}} I = \int_0^{2\pi} \frac{2\,(1+e\cos\gamma)^{-2} + 6Q_\mathrm{R}^\mathrm{max} M_\mathrm{R}^{-1} D_\mathrm{min}^{-2} (1+e)^{-2} \sin\gamma}{2\,(1+e\cos\gamma)^{-2} + 15Q_\mathrm{R}^\mathrm{max} M_\mathrm{R}^{-1} D_\mathrm{min}^{-2} (1+e)^{-2} \sin\gamma} \cos\gamma\,(1+e\cos\gamma)^3\,\mathrm{d}\gamma. \tag{2.82}$$

The quantity

$$Q_\mathrm{R}^\mathrm{max} M_\mathrm{R}^{-1} D_\mathrm{min}^{-2} (1+e)^{-2} \tag{2.83}$$

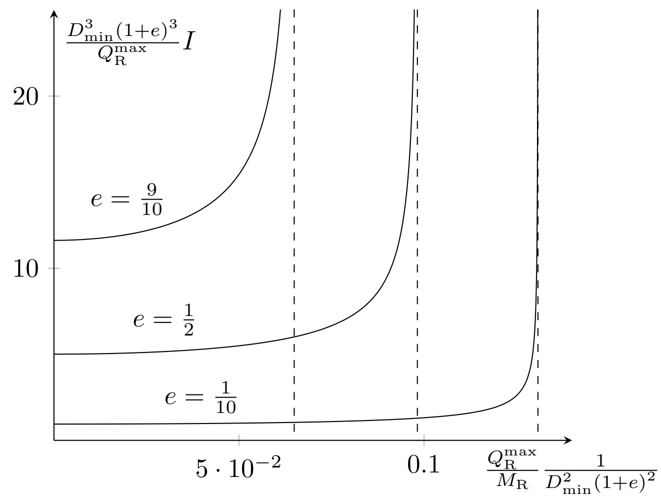

Figure 2.5: Three data points from a numerical investigation regarding the work integral I. If the quadrupole moment causes a singularity in the mass distribution (dashed vertical lines), the work integral diverges. The work integral is always positive.

is positive and has an upper bound of

$$\sqrt{2}\frac{72}{5}\frac{e}{\left(5+\sqrt{1+24e^2}\right)^2\sqrt{6e^2-1+\sqrt{1+24e^2}}}, \tag{2.84}$$

implied by the requirement (2.77) (i.e. if this bound is exceeded, the receiver mass density will blow up). This restriction means that, for a given eccentricity, any possible value of the right hand side of equation (2.82) can be computed numerically. Figure 2.5 shows a few data points from such a search. One may draw the conclusion that the transmitted energy is always positive as long as there is some eccentricity to the orbits, and that the work integral diverges as the quadrupoles approach the singularity condition.

2.7 Evaluation of energy flux density vectors

It is now possible to evaluate the different energy flux density vectors discussed in chapter 1 for the present case of gravitational induction. The vacuum expression for the general flux density vector is

$$\mathbf{S} = N\frac{1}{8\pi}\left(V\nabla\dot{V} - \dot{V}\nabla V\right) - (N-1)\frac{1}{4\pi}V\nabla\dot{V}, \tag{2.85}$$

where V is the full potential of the system in the vacuum outside the gravitating bodies. The three main variants are the Synge/Maxwell[5] flux density ($N=0$), Bondi's flux density ($N=1$), and the

[5]Maxwell did not propose a flux density vector, but this is the one consistent with his energy density expression.

version favored by the investigation in chapter 1 ($N = 2$). Written out explicitly, they are

$$\begin{cases} \mathbf{S}_{\mathrm{M}} = \frac{1}{4\pi} V \nabla \dot{V} \\ \mathbf{S}_{\mathrm{B}} = \frac{1}{8\pi} \left(V \nabla \dot{V} - \dot{V} \nabla V \right) \\ \mathbf{S}_{2} = -\frac{1}{4\pi} \dot{V} \nabla V. \end{cases} \tag{2.86}$$

It should be noted that $\mathbf{S}_2$ can be had as two times $\mathbf{S}_{\mathrm{B}}$ minus one $\mathbf{S}_{\mathrm{M}}$. All the information needed to start making calculations is summarized below.

Using the frame depicted in figure 2.3, take the place of evaluation to be

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{2.87}$$

with the coordinates given by the basis vectors in the referenced figure. The expression for the potential there is

$$V = M_{\mathrm{R}} U_{\mathrm{M}}(\mathbf{r} - \mathbf{R}) + Q_{\mathrm{R}} U_{\mathrm{Q}}(\mathbf{r} - \mathbf{R}) + M_{\mathrm{T}} U_{\mathrm{M}}(\mathbf{r} - \mathbf{T}) + Q_{\mathrm{T}} U_{\mathrm{Q}}(\mathbf{r} - \mathbf{T}), \tag{2.88}$$

with

$$U_{\mathrm{M}}(\mathbf{r} - \mathbf{X}) = -\frac{1}{|\mathbf{r} - \mathbf{X}|}, \tag{2.89}$$

and

$$U_{\mathrm{Q}}(\mathbf{r} - \mathbf{X}) = -\frac{P_2(z/|\mathbf{r} - \mathbf{X}|)}{|\mathbf{r} - \mathbf{X}|^3}. \tag{2.90}$$

The argument of the Legendre polynomial has been transformed from the usual polar angle into Cartesian coordinates, and also simplified using the fact that the orbits are bound to the xy-plane (i.e. the z-components of $\mathbf{r} - \mathbf{X}$ and $\mathbf{r}$ are the same). The positions of the bodies are given by

$$\mathbf{T} = -\frac{m}{M_{\mathrm{T}}} D \begin{pmatrix} \cos\gamma \\ \sin\gamma \\ 0 \end{pmatrix}, \tag{2.91}$$

and

$$\mathbf{R} = \frac{m}{M_{\mathrm{R}}} D \begin{pmatrix} \cos\gamma \\ \sin\gamma \\ 0 \end{pmatrix}, \tag{2.92}$$

where the distance between them is

$$D = D_{\mathrm{min}} \frac{1 + e}{1 + e\cos\gamma}. \tag{2.93}$$

The quadrupole moments are

$$Q_{\mathrm{T}} = -\frac{2 M_{\mathrm{T}} Q_{\mathrm{R}}}{2 M_{\mathrm{R}} - 15 Q_{\mathrm{R}}/D^2} \tag{2.94}$$

(to preserve Kepler orbits), and

$$Q_{\mathrm{R}} = -Q_{\mathrm{R}}^{\mathrm{max}} \sin\gamma \tag{2.95}$$

(to achieve positive energy transfer to the receiver).

The one remaining problem is that the dot derivatives in equation (2.85) are derivatives with respect to time, not with respect to γ. The time dependence of the potential above, however, is ultimately a dependence on γ, so following through differentiation with the chain rule will terminate in a factor of $\dot{\gamma}$. This is resolved by using the conserved angular momentum of the system to write

$$\dot{\gamma} = \frac{l}{D^2}. \tag{2.96}$$

Since the function $D(\gamma)$ is known, the resulting expressions can be evaluated for any value of γ (i.e. at any point along the motion).

This takes care of all the conceptual difficulties. Finding a closed expression for $\mathbf{S}$ in terms of the set of parameters $\{l,\, e,\, D_{\min},\, Q_{\mathrm{R}}^{\max},\, M_{\mathrm{T}},\, M_{\mathrm{R}}\}$ is now an elementary exercise in differentiation, but the large number of terms and parameters, combined with the translated potentials, make it a very tedious one. While it *can* be done by hand it is done much faster and accurately by a machine, so we recommend having a computer algebra system do it for you. A Mathematica notebook that can perform such a task is provided for reference (a text copy is attached as appendix A, and the file is available for download via the reference [5]).

For those who insist on proceeding by hand, there are some simplifications that can be made. Setting the two masses equal cuts down on the number of unique terms significantly. Also, the evaluation simplifies greatly during the periapsis and the apoapsis. At those instances in time, γ, $\dot{D}$, Q_{R}, and Q_{T} all happen to vanish simultaneously. When doing such a calculation, terms proportional to any of those quantities can be thrown away immediately as they are encountered, saving much effort. It has been confirmed, by hand, that the Mathematica notebook gives the correct (or at least the same) result for $\mathbf{S}_{\mathrm{B}}$ during the periapsis, assuming equal masses. We took this as a confirmation that the computer algebra approach works in general, even though it is of course still possible that something has gone wrong somewhere.

The plan was to study the flux of energy during the periapsis. The idea behind this was that energy transfer should be at a maximum then, and that we are therefore more likely to find clues of inductive transfer at this instant. The first part might be true, but it turns out that all expression for gravitational energy flux density vectors are independent of the orbit eccentricity at this point in time (the expressions are not pretty in print, but this is easily checked in the Mathematica notebook). No matter how much energy is transmitted over a cycle, then, the flow during this maximum would look the same.

Two investigations were carried out using the computer algebra approach. For Bondi's flux density vector, a surface integral enclosing one of the bodies can be solved with relative ease. The details are given in the next section. An attempt was also made to illustrate the flow of energy by plotting the vector fields (section 2.7.2). Even though things did not work out as expected, the investigations were carried out during the periapsis. In the coming sections take all flux density fields to be evaluated for $\gamma = 0$, and for equal masses ($M_{\mathrm{R}} = M_{\mathrm{T}} = M$).

2.7.1 Surface integral

Bondi's flux density vector ($N = 1$) is divergence-free in the vacuum, as can be seen by using Poisson's equation to write

$$\nabla\left(V\nabla\dot{V} - \dot{V}\nabla V\right) = V\dot{\rho} - \dot{V}\rho. \tag{2.97}$$

In that case, then, a surface integral that encloses one of the bodies should capture the full throughput of power. The other suggested flux vectors do not share this property, so a corresponding integral does not have the same meaning in those cases.

Surround the receiver by a box-shaped integration surface, with one side lying in the plane $x = 0$ (halfway between the two bodies). Let the surface normal point inward to find the power input. Now grow the five other sides "to infinity", such that the receiver is inside the box of infinite volume, and the transmitter is outside it. It is easy (in Mathematica) to check that the surface integrals over these sides all vanish, leaving only the integral over the plane $x = 0$ to be solved. The surface normal singles out the x-component of the field ($\mathbf{S}_\mathrm{B}$), which is

$$\widehat{\mathbf{x}} \cdot \mathbf{S}_\mathrm{B}\big|_{x=0} = \frac{3}{8\pi}\frac{lM}{D_\mathrm{min}}\left(\frac{MD_\mathrm{min}y}{\left(\frac{1}{4}D_\mathrm{min}^2 + y^2 + z^2\right)^3} + Q_\mathrm{R}^\mathrm{max}\frac{\frac{1}{4}D_\mathrm{min}^2 + y^2 - z^2}{\left(\frac{1}{4}D_\mathrm{min}^2 + y^2 + z^2\right)^4}\right) \tag{2.98}$$

(this can be found by hand or by computer algebra). The surface integral over the plane $x = 0$ is easy to do in polar coordinates;

$$\int_0^\infty \mathrm{d}r \int_0^{2\pi} \mathrm{d}\varphi\, r\hat{\mathbf{x}} \cdot \mathbf{S}_\mathrm{B}\Big|_{x=0}, \tag{2.99}$$

where $y = r\cos\varphi$ and $z = r\sin\varphi$. The integrand, in such coordinates, is

$$\frac{3}{8\pi}\frac{lM}{D_\mathrm{min}}r\left(\frac{MD_\mathrm{min}r\cos\varphi}{\left(\frac{1}{4}D_\mathrm{min}^2 + r^2\right)^3} + Q_\mathrm{R}^\mathrm{max}\frac{\frac{1}{4}D_\mathrm{min}^2 + r^2\left(5\cos^2\varphi - 4\right)}{\left(\frac{1}{4}D_\mathrm{min}^2 + r^2\right)^4}\right). \tag{2.100}$$

Performing the angular integral, it is clear that the first term does not survive. Doing the integrals over what remains gives the result

$$\frac{1}{2}\frac{lMQ_\mathrm{R}^\mathrm{max}}{D_\mathrm{min}^5}, \tag{2.101}$$

which is the expected power,

$$-\frac{\partial W}{\partial Q_\mathrm{R}}\dot{Q}_\mathrm{R} = -\frac{1}{2}\frac{1}{D^3}\left(M - \frac{9}{2}\frac{Q_\mathrm{T}}{D^2}\right)\dot{Q}_\mathrm{R}, \tag{2.102}$$

evaluated at the periapsis. That is, for the values

$$\begin{cases} Q_\mathrm{T} = 0 \\ D = D_\mathrm{min} \\ \dot{Q}_\mathrm{R} = -Q_\mathrm{R}^\mathrm{max}\frac{l}{D_\mathrm{min}^2}. \end{cases} \tag{2.103}$$

It seems very likely that this calculation would work out at any point in time, not just during the periapsis.

2.7.2 Illustrating the flux of energy

Using the referenced Mathematica notebook, some illustrations of the discussed energy flux density fields were produced. Figure 2.6 shows these vector fields evaluated over the orbital plane. The z-components are all zero, so there is no information lost by plotting in two dimensions. We expected to see some indication of energy traveling from transmitter to receiver, but the images are rather hard to interpret. A few remarks could be made about their appearance, however.

The plots for $\mathbf{S}_{\mathrm{M}}$ and $\mathbf{S}_{\mathrm{B}}$ look remarkably similar. The two flux vectors represent opposing views of how the energy is distributed (potential vs. field energy). One ($\mathbf{S}_{\mathrm{B}}$) is also divergence-free in the vacuum, while the other is not. Given these differences, it is a little surprising that the two flows look so alike.

In the field $\mathbf{S}_2$, the flow is zero midway between the bodies. It would appear that no energy is traveling across the space separating the two bodies. This may perhaps be seen as an artifact arising from plotting the flow of two different forms of energy together as one flow. If the flow is decomposed into a flow of potential energy (two times $\mathbf{S}_{\mathrm{B}}$), and a flow of field energy ($-\mathbf{S}_{\mathrm{M}}$), then these happen to cancel in the middle, but energy is, in a sense, flowing across $x = 0$.

There is a clear counter-clockwise rotation in the $\mathbf{S}_2$-field, perhaps related to the motion of the orbits (which is in the same direction). In the other two plots, energy appears to be moving from "behind" each body to the "front" facing the direction of motion. We guess that these plots are all dominated by flow of energy related to the orbital motion, making them hard to interpret.

2.8 Concluding remarks

The main objective of this chapter was to gain a better understanding of Bondi's and McCrea's example of gravitational inductive transfer of energy. It seems clear that the original authors' results where all founded in sound reasoning. Hopefully, the investigation presented here can make the idea more accessible to a wider audience. This main objective shall be considered achieved.

We had also hoped that this could lead to a better understanding of the different options for Newtonian gravitational energy flux density vectors. While there indeed seems to be a connection between inductive transfer and viewing gravitational energy as potential energy, no real conclusions could be drawn. The thought experiment discussed here has a certain ingenious beauty to it, but it might not be the simplest example of inductive transfer. Bondi and McCrea do mention a similar idea wherein an exploding star transfers some energy to a smaller gravitating object, so that could be a candidate for continued study.

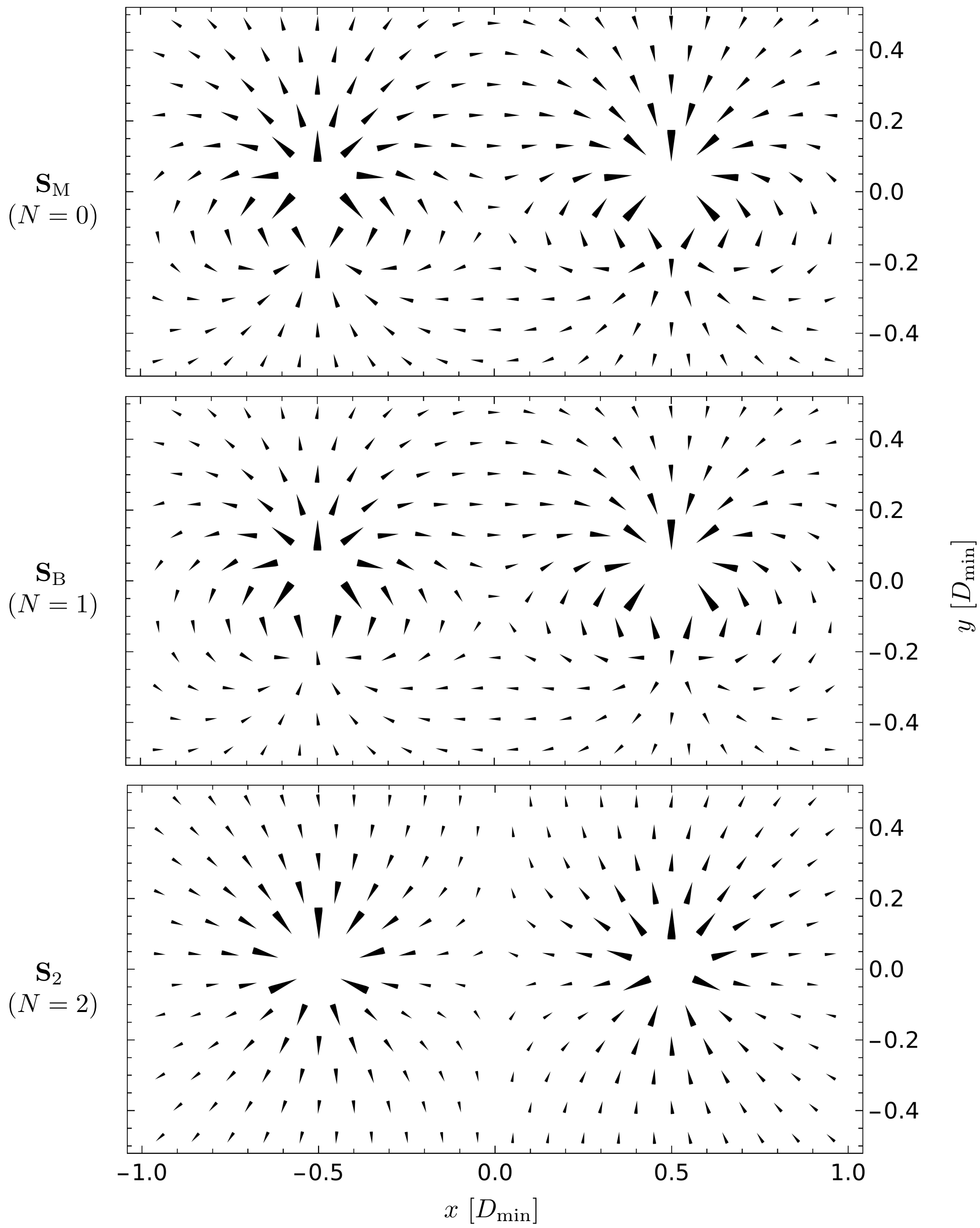

Figure 2.6: Vector plots of the three variants of energy flux density vector for Newtonian gravitation, evaluated during the periapsis. Parameter values are $e = 1/2$, $M_\mathrm{T} = M_\mathrm{T} = 1$, $Q_\mathrm{R}^\mathrm{max} = 1/10$, and $l = 1$ in units such that the gravitational constant has the numerical value one. The transmitter (Tweedledee) is positioned at $x = -1/2$, $y = 0$, and is moving down towards negative y. The receiver (Tweedledum) is positioned at $x = 1/2$, $y = 0$, moving up. Arrow size is indicative of field magnitude (log scale). Very large values near body centers have been excluded.

Chapter 3

Newton-Cartan Theory

We had hoped that the underdetermined local gravitational energy of Newtonian gravitation could somehow be put under a more clarifying light by studying its relationship to energy in general relativity. If Newton's theory is the non-relativistic limit of Einstein's theory, then perhaps the origins of Newtonian local gravitational energy lie in this limiting procedure. One such idea was explored in section 1.3. The non-relativistic limit of a specific energy density appeared to match a Newtonian expression, but with any field energy absent. To dig deeper, one wants to "take the limit" in a more general sense than just studying the weak-field limit of Schwarzschild's vacuum solution.

The best candidate for a systematic treatment of the non-relativistic limit of general relativity is Newton-Cartan theory. In this re-imagining of Newtonian gravitation, the familiar coordinate description is superseded by a covariant formulation, where the trajectories of free falling test particles are not any more given by a potential. Rather, the world lines of freely falling frames are timelike geodesics, and the gravitational field itself vanishes locally. First formulated by Cartan [2] and Friedrichs [7], Newton-Cartan theory has been further developed by Trautman [26, pp. 105–121], Ehlers [4], Künzle [13], and many more.

It is, however, a simple fact of life that time is a finite resource. And as such, time for this project ran out before we could get to any insights regarding energy in Newton-Cartan theory. This final chapter is therefore a short description of the geometrical view of Newtonian gravitation. It is our hope that it can serve as an introduction and reading guide to those who wish to continue down this path. The literature on the subject is not, in general, particularly easy to just pick up and start reading; perhaps because it is not usually taught to physics students. Malament has written a detailed and relatively beginner-friendly chapter on the subject [16, ch. 4]. Sections 3.4 to 3.6 are mostly based on his work and definitions.

The first hurdle for newcomers to get over is the fact that the spacetime geometry of classical physics cannot be described using the usual Riemannian approach to differential geometry. A common way of building up the necessary geometrical concepts of general relativity is to first introduce a measure of distance, the metric. The geodesics and curvature of spacetime then follow from the choice of metric. While pedagogically sound, this is not the most general way of treating curved spaces. In particular, it presupposes the existence of a metric while such an object is not necessary to do differential geometry. Before getting to the more general approach, which here will be referred to as *Cartan theory*, a brief reminder (mostly based on Hartle [11]) of the physics undergraduate version of differential geometry will be given.

3.1 Riemannian geometry in general relativity

In the Riemannian approach, the geodesic equation is derived from the metric. The proper time along a worldline is

$$\tau = \int_A^B \sqrt{-g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma} \frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma}}\, \mathrm{d}\sigma,$$

where $g_{\mu\nu}$ is a coordinate expression of the metric. Finding the geodesics is, then, a variational problem, which leads to the general form of the geodesic equation;

$$\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\tau^2} + \Gamma^\alpha_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = 0. \tag{3.1}$$

The coefficients $\Gamma^\mu_{\rho\sigma}$, the *Christoffel symbols*, are uniquely[1] determined by specifying the metric. The world lines of test particles are then taken to be timelike solutions to this equation.

The *covariant derivative* is a form of differentiation which compares change in vector and tensor fields to that induced by parallel transport (as specified by the metric). This derivative applied to a mixed tensor is itself a tensor with the coordinate basis components

$$\nabla_\alpha A_\beta{}^\gamma = \partial_\alpha A_\beta{}^\gamma + \Gamma^\gamma_{\alpha\mu} A_\beta{}^\mu - \Gamma^\mu_{\alpha\beta} A_\mu{}^\gamma \tag{3.2}$$

(and so on for higher order tensors), in a coordinate basis. By defining the covariant derivative in terms of parallel transport, it follows that the metric is constant;

$$\nabla_\alpha g_{\beta\gamma} = 0. \tag{3.3}$$

The *Riemann curvature tensor* is defined by its action on an arbitrary vector as

$$\left[\nabla_\alpha, \nabla_\beta\right] v^\gamma + R_{\alpha\beta\mu}{}^\gamma v^\mu = 0. \tag{3.4}$$

Its coordinate basis components are given by

$$R_{\alpha\beta\gamma}{}^\delta = \partial_\beta \Gamma^\delta_{\alpha\gamma} - \partial_\alpha \Gamma^\delta_{\beta\gamma} + \Gamma^\delta_{\beta\mu} \Gamma^\mu_{\alpha\gamma} - \Gamma^\delta_{\alpha\mu} \Gamma^\mu_{\beta\gamma}. \tag{3.5}$$

This useful object encodes the curvature of spacetime, but remember; the covariant derivative, and therefore the curvature of spacetime, was specified by assuming a certain metric.

The geodesics, covariant derivative, and curvature tensor all form a neat package relating back to the initial choice of metric. Specifying how distances are measured thus leads to a natural understanding of these concepts, but this intuitive way of doing things shall now be abandoned. Using ideas developed by Élie Cartan, the notions of curvature and differentiation can be generalized to non-metric spaces. To understand the geometry of Newtonian theory, it is necessary to first understand these generalizations.

3.2 Cartan theory

As mentioned, there does not need to be such an object as a metric in order to do differential geometry. Instead, one starts with just the spacetime manifold (no inner product implied) and

[1] Assuming they are taken to be symmetric in the two lower indices. Setting the antisymmetric part, the *torsion*, to zero does not affect the geodesic equation.

directly introduces a *connection*, specifying the relationship between nearby tangent spaces. In this context, the coefficients $\Gamma^{\alpha}_{\beta\gamma}$ are usually called *connection coefficients*. Because the connection, the covariant derivative, and the connection coefficients are all closely related, these terms are often used interchangeably. In this text, specifying any (and thus all) of these shall be referred to as a choice of connection. It shall also be permitted to refer to the covariant derivative or the coefficients as the connection. Importantly, no notion of length has been introduced, but the connection now defines geodesics, covariant differentiation, and curvature (via equations (3.1), (3.2), and (3.4), respectively) independently of the existence of a metric.

Before proceeding, a result concerning connection coefficients is needed. A transformation rule can be derived from the claim that the expression $\nabla_a t_\beta{}^\gamma$ gives the components of a tensor. Changing coordinates from x^α to $x'^\alpha(x^\alpha)$, the first term on the right in equation (3.2) does not behave like a tensor. The only way this can work out is having the connection coefficients transform as

$$\Gamma'^{\alpha}_{\beta\gamma} = \frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x^\nu}{\partial x'^\beta}\frac{\partial x^\rho}{\partial x'^\gamma}\Gamma^{\mu}_{\nu\rho} + \frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial^2 x^\mu}{\partial x'^\beta \partial x'^\gamma} \tag{3.6}$$

to compensate. This also demonstrates the standard claim that these coefficients are not tensors.

In contrast with the Riemannian approach, there is now no privileged way of measuring change in tensor fields. Assuming the coefficients $\Gamma^{\alpha}_{\beta\gamma}$ satisfy equation (3.6), then so will the set $\widehat{\Gamma}^{\alpha}_{\beta\gamma}$ if the difference between them are the components of a tensor. That is, if

$$\Gamma^{\alpha}_{\beta\gamma} = \widehat{\Gamma}^{\alpha}_{\beta\gamma} + C^{\alpha}{}_{\beta\gamma}, \tag{3.7}$$

and

$$C'^{\alpha}{}_{\beta\gamma} = \frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x^\nu}{\partial x'^\beta}\frac{\partial x^\rho}{\partial x'^\gamma}C^{\mu}{}_{\nu\rho}, \tag{3.8}$$

then $C'^{\alpha}{}_{\beta\gamma}$ will appear identically on both sides of equation (3.6) when substituting $\widehat{\Gamma}^{\alpha}_{\beta\gamma} + C^{\alpha}{}_{\beta\gamma}$ for $\Gamma^{\alpha}_{\beta\gamma}$. So, the coefficients $\widehat{\Gamma}^{\alpha}_{\beta\gamma}$ transform as connection coefficients should. The connection with this set of coefficients is a valid connection, and thus a derivative given by

$$\widehat{\nabla}_\alpha A_\beta{}^\gamma = \nabla_\alpha A_\beta{}^\gamma + C^{\mu}{}_{\alpha\beta}A_\mu{}^\gamma - C^{\gamma}{}_{\alpha\mu}A_\beta{}^\mu \tag{3.9}$$

is also valid for performing covariant differentiation. This derivative also satisfies a corresponding equation like (3.2) (replace ∇ with $\widehat{\nabla}$, and Γ with $\widehat{\Gamma}$).

It is of particular interest that a trajectory which is curved according to one connection can be a straight line according to another. A particle with four-velocity ξ^α has the acceleration

$$\xi^\mu \nabla_\mu \xi^\alpha = a^\alpha. \tag{3.10}$$

Now use equation (3.9) to write

$$\xi^\mu \widehat{\nabla}_\mu \xi^\alpha + \xi^\mu C^{\alpha}{}_{\mu\nu}\xi^\nu = a^\alpha, \tag{3.11}$$

then choose

$$C^{\alpha}{}_{\mu\nu} = \sigma_\mu \sigma_\nu a^\alpha \tag{3.12}$$

where

$$\sigma_\mu \xi^\mu = 1. \tag{3.13}$$

The acceleration now vanishes according to $\widehat{\nabla}_\alpha$;

$$\xi^\mu \widehat{\nabla}_\mu \xi^\alpha = 0. \tag{3.14}$$

Cartan himself discovered early on that the Newtonian gravitational field can be absorbed by the connection in this way [2]. Before moving on to geometrization of Newtonian gravitation, some more structure is needed, however (see section 3.4).

To reconnect with the familiar Riemannian geometry, introduce an invertible symmetric tensor field with components $g_{\alpha\beta}$. For a given such field, a metric, there is only one torsion-free (i.e. symmetric) connection with the property

$$\nabla_\alpha g_{\beta\gamma} = 0; \tag{3.15}$$

the *Levi-Civita connection.* This condition on the connection is known as *metric compatibility.* To see that the compatible connection is unique, write out metric compatibility as

$$\nabla_\alpha g_{\beta\gamma} = \partial_\alpha g_{\beta\gamma} - \Gamma^\mu_{\alpha\beta} g_{\mu\gamma} - \Gamma^\mu_{\alpha\gamma} g_{\beta\mu} = 0. \tag{3.16}$$

By cyclical permutation of the indices (and use of index symmetries in $\Gamma^\alpha_{\beta\gamma}$ and $g_{\alpha\beta}$) this can be solved for the connection coefficients in terms of the components of the metric, giving the formula

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\mu} \left(\partial_\beta g_{\mu\gamma} + \partial_\gamma g_{\alpha\mu} - \partial_\mu g_{\alpha\gamma} \right). \tag{3.17}$$

These connection coefficients are, of course, the Christoffel symbol of the previous section. The Riemannian geometry of general relativity is, then, a special case of Cartan theory, where the connection has been required to be compatible with a certain metric.

3.3 Abstract index notation

In previous sections, care was taken not to confuse claims about the components of tensors in a coordinate basis with claims about the vectors and tensors themselves. Specifically, Ricci calculus was used for expressing components. While such notation is very useful for doing calculations in coordinate systems, it is cumbersome not to have a notation for expressing invariant truths about the tensors and vectors themselves. To this end, *abstract index notation* will be used in the following sections.

Roger Penrose invented the abstract index notation as a means of bridging the gap between the purely abstract (that is, without reference to coordinates or frames) nature of differential geometry and the concrete, but useful, notation used by many phycisists. The interested reader may want to read a more technical account (for example [20, p. 135], or [16, pp. 24–35]). Here it is enough to lay down the basic rules.

In contrast with the traditional Ricci notation, an abstract index should not be interpreted as labeling components, but rather as labeling objects by what space they belong in. Given a tangent space V at a point, v^a is a vector in V, and v_a is a linear map in its dual space, V*. Contractions like $u_m v^m$ are not to be interpreted as a sum over indices, but simply as the image of v^a under u_a.

An order two mixed tensor, an element of the tensor product V $\otimes$ V*, is similarly written as $A^a{}_b$. Here, it was necessary to introduce two unique labels to distinguish the two arguments of the corresponding bilinear map (the order of indices therefore needs to be retained). It is clear, then, that $A^m{}_m$ should be interpreted as tensor contraction and that $v^a u_b$ is an outer product.

Adopting this convention creates a need to distinguish between abstract and concrete indices. In the following, Roman letters, starting from a, will be used for abstract indices, and Greek letters, starting from α, will be used for coordinate expressions. Contractions will be made more distinct by starting the labeling from m for abstract indices, and from μ for summation over concrete indices. Of course, there also needs to be an exception to these simple rules; the letters i, j, and k, although Roman, are sometimes used in coordinate expressions to denote a restriction to spatial components. With these conventions, it should be easy to see what kind of statement one is dealing with at a glance.

Fully covariant index expressions look identical in concrete and abstract indices. Some of the expressions already encountered can therefore easily be translated into abstract form. For example, equation (3.9) is expressed as

$$\widehat{\nabla}_a A_b{}^c = \nabla_a A_b{}^c + C^m{}_{ab} A_m{}^c - C^c{}_{am} A_b{}^m. \tag{3.18}$$

Similarly, the curvature tensor of a connection is given by

$$[\nabla_a, \nabla_b]\, v^d + R_{abm}{}^d v^m = 0 \tag{3.19}$$

in abstract notation.

3.4 Classical spacetime

In Malament's rendering of Newton-Cartan theory, a distinction is made between just expressing Newton's theory as differential geometry, and the geometrized theory, usually called Newton-Cartan theory. In the former, there is still a scalar field, the potential, determining the acceleration of particles, while in the latter, this field is absorbed by the connection in order to describe free falls as geodesics. Both of these can be described as a structure called a *classical spacetime*. An account of its basic workings follows below. This text shall not be very formal, readers may want to refer to pages 249 to 266 of Malament's book [16] for more details.

A classical spacetime, denoted by $(M, t_a, h^{ab}, \nabla_a)$,[2] consists of four objects; the four-dimensional manifold of point events (M), two fields for measuring duration and length (t_a and h^{ab}, respectively), and a torsion-free connection (∇_a) with associated, through equation (3.19), curvature tensor ($R_{abc}{}^d$). For any classical spacetime, the signature of the spatial "metric" is $(0, 1, 1, 1)$, and the conditions

$$h^{am} t_m = 0 \tag{3.20}$$

(referred to as *metric orthogonality*),

$$\nabla_a t_b = 0, \quad \text{and} \quad \nabla_a h^{bc} = 0 \tag{3.21}$$

(metric compatibility) shall be satisfied. It is of great importance to note that neither t_a nor h^{ab} are metric tensors in the usual sense. A metric is usually required to be invertible. The terminology used here is, however, convenient, and seems quite common.

[2]The structure considered here is actually what Malament calls a *temporally orientable* classical spacetime with a chosen temporal orientation, but the more general structures, $(M, t_{ab}, h^{ab}, \nabla_a)$, are not of interest here.

Contrary to the Riemannian case discussed earlier, this weaker form of metric compatibility does not fix the connection uniquely. Given the two metric fields, t_a and h^{ab}, and a compatible connection ∇_a, another connection $\widehat{\nabla}_a$ is also compatible with both metrics if $C^a{}_{bc}$ in equation (3.18) is of the form

$$C^a{}_{bc} = 2h^{an}t_{(b}\kappa_{c)n} \tag{3.22}$$

(with κ_{ab} an antisymmetric field). Two otherwise identical classical spacetimes can be endowed with different connections, and therefore different curvature. This freedom is what will allow both the traditional and geometrized theories to be realized as classical spacetimes, differing only in the choice of connection.

To make the notation more compact, a variation on the common index-raising convention is often used. There is no proper metric available to do this, so covariant indices are instead raised with the spatial metric according to the rule

$$h^{am}A_m{}^b = A^{ab}. \tag{3.23}$$

Because this operation is not invertible, it is best to avoid introducing a convention for lowering indices.

A vector v^a is given the *duration*

$$t_m v^m,$$

and vectors are classified into three categories depending on the result of this operation. They are called *spacelike* if the duration is zero, and *timelike* if it is not. Further, timelike vectors are *future-directed* if the duration is positive and *past-directed* if it is negative. These terms can be extended to define spacelike hypersurfaces and timelike curves in a natural way; tangent vectors to a spacelike hypersurface are spacelike at every point of the surface and tangent vectors to timelike curves are timelike at every point on the curve. The tangent field to a timelike curve, parametrized by its temporal length, is of unit duration. Four-velocities are also normalized in the same way.

The assumptions so far guarantee the existence of a scalar universal time function. The compatibility condition for t_a immediately gives that $\nabla_{[a}t_{b]} = 0$ which, by the Poincaré lemma, implies that there is a global[3] function t such that

$$t_a = \nabla_a t. \tag{3.24}$$

The existence of a universal time function is in keeping with Newton's world view, and, as shall be seen in the next section, it is necessary to choose this function as a coordinate to recover the familiar coordinate expressions of his theory.

A notion of duration has been established, but how does the spatial metric assign lengths to spacelike vectors? This is achieved by considering any covector σ_a which obeys

$$\mu^a = h^{am}\sigma_m \tag{3.25}$$

and then taking the length of μ^a to be

$$\sqrt{h^{mn}\sigma_m\sigma_n} \tag{3.26}$$

if μ^a is spacelike. Note that spacelike vectors have zero duration (by definition), but timelike vectors have not been assigned any spatial length.

[3]This requires that the manifold is simply connected.

The covector σ_a of the previous paragraph is guaranteed to exist if μ^a is spacelike, but it is not unique. The length assigned to μ^a, however, is. There is a freedom of adding a covector to σ_a as long as it does not contribute in the contraction of equation (3.25), but this covector must also vanish in equation (3.26) (because the spatial metric is symmetric) and so, the length of μ^a is the same for all choices of σ_a.

3.5 Newton's theory in geometrical language

The traditional theory of Newtonian gravity, where free falls are given by a potential, can be stated as a classical spacetime. In this case, the derivative operator will be assumed to be flat, meaning that the associated curvature tensor vanishes everywhere. Given a mass distribution ρ, the gravitational potential satisfies Poisson's equation

$$\nabla_m \nabla^m \phi = 4\pi\rho, \tag{3.27}$$

and a test particle with the four-velocity ξ^a has the acceleration

$$\xi^m \nabla_m \xi^a = -\nabla^a \phi. \tag{3.28}$$

This is Malament's version of Newtonian gravitation, but it is not so obvious that it is identical to the familiar theory. To see that it is, one should recover the coordinate expressions.

Trautman has shown [26, pp. 113–114] that metric compatibility and orthogonality, the signature condition on the spatial metric, and the existence of a universal time function all together imply that there exists a choice of coordinates, x^α, such that

$$\begin{cases} x^0 = t, \\ \Gamma^\alpha_{\beta\gamma} = 0, \\ h^{0\alpha} = h^{\alpha 0} = 0, \\ h^{ij} = \delta^{ij}. \end{cases} \tag{3.29}$$

Here, and in the rest of this section, the indices i and j, run only over the spatial components $(1, 2, 3)$. These coordinates are the usual inertial Cartesian coordinates, in which the equation of motion takes the form

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} x^i(t) = -\delta^{ij}\partial_j \phi \tag{3.30}$$

(where $x^j(t)$ is the trajectory of a particle), and where Poisson's equation becomes

$$\delta^{ij}\partial_i\partial_j\phi = 4\pi\rho. \tag{3.31}$$

The full argument from Trautman will not be repeated here, but it is not too hard to see how the properties (3.29) of this coordinate system gives the usual equation of motion, starting from Malament's version above. Given Trautman's coordinates, where the connection coefficients all vanish, the coordinate basis expression for the left hand side of equation (3.28) is (c.f. the geodesic equation)

$$\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}t^2}, \tag{3.32}$$

and the right hand side is

$$-\, h^{\alpha\mu}\partial_\mu \phi. \tag{3.33}$$

For $\alpha = 0$ this gives

$$\frac{\mathrm{d}^2 x^0}{\mathrm{d}t^2} = 0, \tag{3.34}$$

consistent with the choice $x^0 = t$. For $\alpha = i$, the sought after result is immediate.

3.6 Geometrization

Moving on to the geometrized theory is now relatively easy. The reader may remember the trick used to make the acceleration field disappear in section 3.2. In that case, no restriction was placed on the geometrizing connection. It was claimed earlier that the background structure of geometrized Newtonian gravitation is a classical spacetime (as defined in section 3.4), so now the objective is to find a geometrizing connection that retains metric compatibility. Except for that detail, the argument is the same; find a tensor $C^a{}_{bc}$ such that the derivative $\widehat{\nabla}_a$, given by equation (3.18), absorbs the acceleration field in the flat derivative equation of motion (3.28).

The freedom available for choosing a compatible connection is given by equation (3.22). Now, let the antisymmetric field κ_{ab} of that equation be

$$\kappa_{ab} = -t_{[a}\nabla_{b]}\phi. \tag{3.35}$$

By a short calculation it is found that

$$C^a{}_{bc} = -t_b t_c \nabla^a \phi. \tag{3.36}$$

Define the geometrizing connection ($\widehat{\nabla}_a$) as that which has the action (3.36) relative to the flat connection (∇_a).[4] Rewriting the equation of motion using this connection gives

$$\xi^m \widehat{\nabla}_m \xi^a - \xi^m t_m t_n \nabla^a \phi \xi^n = -\nabla^a \phi. \tag{3.37}$$

The four-velocity ξ^a is a unit timelike vector, so the terms involving the potential cancel and the result is

$$\xi^m \widehat{\nabla}_m \xi^a = 0. \tag{3.38}$$

Particle trajectories (influenced only by gravity) are geodesics according to this curved connection, which also forms a classical spacetime with the original manifold and metric fields. This geometrization is unique in that it is the only solution κ_{ab} which cancels the given gravitational field.

The curvature tensor associated with $\widehat{\nabla}_a$ satisfies the equation,

$$\widehat{R}_{ab} = 4\pi\rho t_a t_b \tag{3.39}$$

(where the Ricci tensor is $\widehat{R}_{anb}{}^n$ as usual). This is the Newtonian version of Einstein's field equation. It expresses how spacetime is curved by the matter contents of the universe. The quest for a geometrized Newtonian gravitation has thus reached a satisfying conclusion. The curvature produced in this way does, however, behave in a rather peculiar way, which will be the topic of the next section.

[4] See equation (3.18).

3.7 Space is flat

It is a surprising fact that the geometry of classical spacetime, even in the geometrized theory, dictates that space is completely flat. The goal was to recast curved particle trajectories as straight lines in a curved spacetime, but in a sense this resulted only in a time-directed curvature. The coordinate-free argument is rather too advanced for a detailed account in this text, but the main points are as follows. The spatial metric h^{ab} induces a three-dimensional metric on each spacelike hypersurface. This is a metric in the usual sense (i.e. not degenerate). It turns out that the property

$$\widehat{R}^{ab} = 0, \tag{3.40}$$

which follows from the geometrized Newtonian field equation (3.39), implies that this three-dimensional metric is just δ^{ij}. That is, each simultaneous space slice is endowed with the common Euclidean metric. It is again appropriate to refer to Malament's book, where such an argument can be found [16, pp. 260–262]. Here, a simpler demonstration using coordinate expressions will suffice. The following argument is adapted from [18, p. 291].

In section 3.5, it was claimed that one can find coordinates such that the conditions

$$\begin{cases} x^0 = t, \\ \Gamma^\alpha_{\beta\gamma} = 0, \\ h^{0\alpha} = h^{\alpha 0} = 0, \\ h^{ij} = \delta^{ij} \end{cases} \tag{3.41}$$

all hold. They are inertial coordinates in the sense that the connection coefficients of the flat connection vanish, but what about those of the curved, geometrizing connection? Going all the way back to equation (3.7), and setting $\Gamma^\alpha_{\beta\gamma}$ to zero, one finds

$$\widehat{\Gamma}^\alpha_{\beta\gamma} = \partial_\beta t\, \partial_\gamma t\, h^{\alpha\mu} \partial_\mu \phi. \tag{3.42}$$

If α refers to the time component, then the coefficient is zero. For spatial components, the result is

$$\widehat{\Gamma}^i_{00} = \delta^{ij} \partial_j \phi \tag{3.43}$$

and

$$\widehat{\Gamma}^i_{jk} = 0, \tag{3.44}$$

because $\partial_j t = 0$ (these coordinates where chosen such that t is constant along the spatial directions [26, p. 113]).

By equation (3.5), the components of the corresponding curvature tensor ($\widehat{R}_{\alpha\beta\gamma}{}^{\delta}$) are, then, zero for $\delta = 0$ and for $\gamma \neq 0$. This leaves the possible non-zero components

$$\widehat{R}_{\alpha\beta 0}{}^{j} = \delta_{\alpha 0} \partial_\beta \left(\delta^{jk} \partial_k \phi \right) - \delta_{\beta 0} \partial_\alpha \left(\delta^{jk} \partial_k \phi \right). \tag{3.45}$$

Further, these vanish when α and β are equal, and when α and β are both spatial. The remaining non-zero components of the curvature tensor are

$$\widehat{R}_{0i0}{}^{j} = -\widehat{R}_{i00}{}^{j} = \delta^{jk} \partial_i \partial_k \phi. \tag{3.46}$$

Parallel transport of a vector A^α along an infinitesimal square loop gives the change

$$\delta A^\alpha = \widehat{R}_{\mu\nu\rho}{}^{\alpha} u^\mu v^\nu A^\rho, \tag{3.47}$$

where u^α and v^α are the sides of the loop. The change in the time component is clearly zero,

$$\delta A^0 = 0, \tag{3.48}$$

and the other components change according to

$$\delta A^i = \left(u^0 v^k - u^k v^0\right) \delta^{ij} \partial_k \partial_j \phi A^0. \tag{3.49}$$

If the loop lies in a hypersurface of constant time, then u^0 and v^0 are both zero, which infers no change during parallel transport. Because this holds over all of spacetime, every spacelike hypersurface is flat. If the loop traverses some such surfaces in the time direction, the transported vector may come back changed; curvature resides in the time direction.

3.8 Newton-Cartan theory and general relativity

It is often said that Newtonian gravitation should be considered a non-relativistic limit of general relativity. One well-known example was used in chapter 1; the Schwarzschild metric reproduces Newtonian trajectories in the weak-field limit. While looking at limits of specific solutions to Einstein's field equation is interesting in its own right, the existence of a geometrical formulation of Newtonian theory suggests a more general sense in which Newton's theory can be seen as the non-relativistic limit of the Einstein's theory. Work in this area has been done by Künzle [13] and Ehlers [4], among others. Before getting into the abstract approach to taking the limit, a clarifying, frame-dependent, example is discussed below.

The following way of looking at the matrix representation of a Lorentzian metric makes it seem not so far-fetched that, from an observers perspective, a large value of c could make the geometry of general relativity resemble that of Newton-Cartan theory. Consider a local inertial frame in general relativity. There, the metric locally takes the form

$$[g_{ab}] = \mathrm{diag}\left(-c^2, 1, 1, 1\right), \tag{3.50}$$

and its inverse looks like

$$\left[g^{ab}\right] = \mathrm{diag}\left(-\frac{1}{c^2}, 1, 1, 1\right). \tag{3.51}$$

Now introduce the temporal metric

$$t_{ab} = -\frac{1}{c^2} g_{ab}. \tag{3.52}$$

The general relativistic measure of proper time,

$$\mathrm{d}\tau = \sqrt{-\frac{1}{c^2} g_{nm} \mathrm{d}x^n \mathrm{d}x^m}, \tag{3.53}$$

(for timelike $\mathrm{d}x^a$) then becomes

$$\mathrm{d}\tau = \sqrt{t_{nm} \mathrm{d}x^n \mathrm{d}x^m}. \tag{3.54}$$

For measuring proper distance (for spacelike $\mathrm{d}x_a$), use

$$\mathrm{d}l = \sqrt{g^{nm}\mathrm{d}x_n\mathrm{d}x_m}. \tag{3.55}$$

Inversion of the metric can now be written as

$$t_{am}g^{mb} = -\frac{1}{c^2}\delta^b_a. \tag{3.56}$$

Taking the limit $c \to \infty$ produces two orthogonal and degenerate "metrics", with the matrix representations

$$\left[g^{ab}\right] = \mathrm{diag}\,(0, 1, 1, 1) \tag{3.57}$$

and

$$[t_{ab}] = \mathrm{diag}\,(1, 0, 0, 0)\,. \tag{3.58}$$

Measuring time and space according to the above prescriptions,[5] they are now separate phenomena! In previous section g^{ab} was given the name h^{ab}, and it was assumed that one can write $t_{ab} = t_a t_b$ for some vector field t_a.

Ehlers proposed a *frame theory* for studying the non-relativistic limit. This is a set of axioms which give those of general relativity for finite values of c, and a geometrized Newtonian theory when $1/c^2$ is set to zero [4]. Importantly, the geometrized classical theory thus received is not identical to the "traditional" Newton-Cartan theory discussed earlier. The curvature tensor found when geometrizing a Newtonian potential (section 3.6) has the property [16, p. 268, 4, pp. 8–9]

$$\widehat{R}_{ab}{}^{cd} = 0. \tag{3.59}$$

While other symmetries of the curvature tensor, and the geometrized field equation (3.39), follow from Ehlers' geometrical limiting procedure, this one does not. This extra property can be interpreted as a "law of existence of absolute rotation" [4, p. 9], meaning that it is possible to find a frame that sees no rotational inertial forces. The existence of such frames is, then, a special property of Newtonian gravitation, not of the non-relativistic limit of general Lorentz spacetimes.

3.9 Continued study

As has been mentioned, time ran out for this project before any real benefits could be reaped from the study of Newton-Cartan theory. It is our hope that this chapter, such as it is, can help future students or researchers get started with the geometrical view of Newtonian gravitation, and continue where we have left off. This chapter will conclude with two suggestion on where to go next.

Here, the focus has been on the motion of test particles. An obvious next step is to study the extension of Newton-Cartan theory to continuous matter fields. Work in this area has been done by Künzle [13, p. 448], and Malament has also written some helpful paragraphs on the subject [16, p. 266]. If their work is properly understood, then perhaps some clarifying insights can be reached about the status of local energy in Newtonian gravitation.

[5] In the absence of a proper metric, there is no longer a bijection between vectors and covectors. It turns out that the length assigned to a spacelike vector is unique anyway, see equations (3.25) and (3.26).

If we dare to dream big, then we would hope that some sort of energy in general relativity could be associated with Newtonian gravitational energy in the non-relativistic limit using the tools presented here. One such candidate was studied in chapter 1; the matter energy $T^{am}\xi_m$ of a static Lorentz spacetime with timelike Killing field ξ^m. There are no Killing vectors in Newton-Cartan theory, but some of their properties are shared with the *rigid* vectors (defined by $\nabla^{(a}\xi^{b)} = 0$) of a classical spacetime [16, p. 271]. Understanding the relationship between Killing vectors and rigid vectors could be another topic for continued study.

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Appendix A

Computer Model of Newtonian Gravitational Induction

The following Mathematica notebook sets up the necessary definitions to compute the Newtonian gravitational energy flux density field of Bondi's and McCrea's gravitational induction experiment (see chapter 2). Some example computations are also given. The notebook is available for download via the reference [5]. The code was written and tested in Mathematica 12.

A.1 Definitions

In[1]:=
```
(*Potential of the whole system.*)
(*The vector r is the place of evaluation*)
V[r_,t_]:= Mt Um[r + m/Mt dvec[t]] + Qt[t] Uq[r + m/Mt dvec[t]]
      + Mr Um[r - m/Mr dvec[t]] + Qr[t] Uq[r - m/Mr dvec[t]]
```

In[2]:=
```
(*Reduced mass.*)
m = Mt Mr/(Mt + Mr);
```

In[3]:=
```
(*Potential per monopole moment*)
Um[x_]:= -1/norm[x]
```

In[4]:=
```
(*Potential per quadrupole moment, x[[3]]/norm[x] = Cos[theta] if
      x[3]=z.*)
Uq[x_]:= -LegendreP[2,x[[3]]/norm[x]]/norm[x]^3
```

In[5]:=
```
(*Norm of a vector. Mathematica fails to differentiate its own
      routine Norm[x], so it had to be reimplemented.*)
norm[x_]:= Sqrt[x[[1]]^2 + x[[2]]^2 + x[[3]]^2]
```

In[6]:=
```
(*Transmitter quadropole moment.*)
(*This is the condition for Kepler orbits.*)
Qt[t_]:= -2 Mt/(2 Mr - 15 Qr[t]/d[t]^2) Qr[t]
```

In[7]:=
```
(*Receiver quadrupole moment.*)
(*This choice gives positive net transfer to receiver if Qrmax >0.*)
Qr[t_]:= -Qrmax Sin[gamma[t]]
```

In[8]:=
```
(*Separation vector between transmitter and receiver.*)
dvec[t_]:= d[t] {Cos[gamma[t]],Sin[gamma[t]],0}
```

In[9]:=
```
(*Length of the separation vector.*)
(*This makes the trajectory an ellipse.*)
(*The parameter e is the orbit eccentricity*)
d[t_]:= dmin (1 + e)/(1 + e Cos[gamma[t]])
```

In[10]:=
```
(*The time derivative gamma'[t] is known in therms of the conserved
      angular momentum l.*)
(*Defining the derivative in this way, it is automatically substituted
      as the symbol gamma'[t] is encountered by Mathematica.*)
gamma'[t_]:= l/d[t]^2
```

In[11]:=
```
(*The general vacuum energy flux density vector for Newtonian
      gravitation*)
S[r_,t_]:= n/(8Pi) (V[r,t] Grad[D[V[r,t],t],r]
      -D[V[r,t],t] Grad[V[r,t],r])
      -(n - 1)/(4Pi) V[r,t] Grad[D[V[r,t],t],r]
```

A.2 Getting usable expressions

In[12]:=
```
(*Perform the necessary differentiations (w.r.t time and space) by
      evaluating for an input vector.*)
(*This provides an expression depending on dmin, gamma[t], Mr, Mt, e,
      l, Qrmax, and n.*)
Seval = S[{x,y,z},t];
```

In[13]:=
```
(*Choose some parameter values.*)
params = {Mr → 1,Mt → 1,dmin → 1,Qrmax → 1/10,l → 1,e → 1/2};
```

In[14]:=
```
(*It is now possible to compute a result. For example, at the
      periapsis (gamma[t]=0).*)
Speriapsis = Seval//.params//.gamma[t] → 0;
```

In[15]:=
```
(*The vector value at the origin is*)
Speriapsis//.{x → 0,y → 0,z → 0}
```

Out[15]= $\{-\frac{24\ (-1+n)}{5\ \pi}+\frac{12\ n}{5\ \pi},0,0\}$

In[16]:=
```
(*The result during the periapsis is independent of orbit
      eccentricity.*)
D[Speriapsis,e]
```

Out[16]=
```
{0,0,0}
```

A.3 Plotting the flux fields

In[17]:=
```
(*The command below plots the three main types of energy flux density
      during the periapsis, over the plane z=0.*)
scaling = "Log";
clip = 500;
points = 20;
plotrange = 1;
options = {VectorScaling → scaling, VectorColorFunction → None,
      VectorStyle → Black, VectorPoints → points,
      VectorRange → {0,clip}, ClippingStyle->None};
(*The case n=0, Bondi's energy flux.*)
S0peri = VectorPlot[Speriapsis[[{1,2}]]//.{n → 0, z → 0},
      {x,-plotrange,plotrange}, {y,-plotrange,plotrange},
      Evaluate[options], PlotLabel → "n=0 'Synge/Maxwell'"];
S1peri = VectorPlot[Speriapsis[[{1,2}]]//.{n  → 1,z → 0},
      {x,-plotrange,plotrange}, {y,-plotrange,plotrange},
      Evaluate[options], PlotLabel → "n=1 'Bondi'"];
S2peri = VectorPlot[Speriapsis[[{1,2}]]//.{n → 2, z→0},
      {x,-plotrange,plotrange}, {y,-plotrange,plotrange},
      Evaluate[options], PlotLabel → "n=2 'Ours/Lynden-Bell & Katz'"];
GraphicsRow[{S0peri,S1peri,S2peri}, ImageSize → Full]
(*The results are complicated and hard to interpret.*)
```

Output has been excluded. See figure 2.6, which is similar.

A.4 Surface integral

In[24]:=
```
(*Bondi's flux vector is divergence-free in the vacuum.*)
Div[Speriapsis//.n → 1,{x,y,z}]//Simplify
```

Out[24]=
```
0
```

In[25]:=
```
(*Let's investigate Bondi's flux field over the plane x=0 during the
      periapsis.*)
(*For simplicity, the masses are set equal.*)
(*This result has been confirmed by hand.*)
S1perix0 = Seval//.{n → 1,Mt → Mr,x → 0,gamma[t] → 0}//Simplify
```

Out[25]=

$$\{24\ l\ Mr\ (dmin^2\ Qrmax + dmin^3\ Mr\ y + 4\ Qrmax\ (y^2 - 4\ z^2) + 4\ dmin\ Mr\ y\ (y^2 + z^2))/\ (dmin\ \pi\ (dmin^2 + 4\ (y^2 + z^2))^4),0,0\}$$

In[26]:=
```
(*The integral over this plane should capture all transmitted energy.*)
```

In[28]:=
```
(*Convenient change of variables*)
tmp = r S1perix0//.{y^2 + z^2 → r^2,y → r Cos[phi],
      z → r Sin[phi]}//Simplify
```

Out[28]=
```
{24 l Mr r (dmin Mr r (dmin^2 + 4r^2) Cos[phi] + Qrmax (dmin^2 6 r^2 +
      10 r^2 Cos[2phi]))/(dmin π (dmin^2 + 4 r^2)^4),0,0}
```

In[29]:=
```
intphi = Integrate[tmp[[1]],phi]//Simplify  (*Integrate w.r.t phi*)
```

Out[29]=
```
24 l Mr r (Qrmax (dmin^2 - 6 r^2) phi + dmin Mr r (dmin^2 +
      4 r^2) Sin[phi] + 5 Qrmax r^2 Sin[2 phi])/(dmin π (dmin^2 + 4r^2)^4)
```

In[30]:=
```
intphi//.phi → 0 (*The lower limit does not contribute*)
```

Out[30]= 0

In[31]:=
```
intphi2pi = intphi//.phi → 2Pi (*so the definite integral is*)
```

Out[31]=
$$\frac{48\ l\ Mr\ Qrmax\ r\ (dmin^2 - 6\ r^2)}{dmin\ (dmin^2 + 4\ r^2)^4}$$

In[32]:=
```
intr = Integrate[intphi2pi,r] (*integrate w.r.t r*)
```

Out[32]=
$$-\frac{l\ Mr\ Qrmax\ (dmin^2 - 36\ r^2)}{2\ dmin\ (dmin^2 + 4\ r^2)^3}$$

In[33]:=
```
Limit[intr,{r → Infinity}] (*The contribution at infinity vanishes*)
```

Out[33]= 0

In[34]:=
```
-intr//.r → 0 (*so the result is*)
```

Out[34]=
$$\frac{l\ Mr\ Qrmax}{2\ dmin^5}$$

In[35]:=
```
(*We should also check that there is no contribution from surfaces at
      infinity.*)
Limit[(x^2+y^2+z^2) Speriapsis[[1]]//.n → 1,x → ∞]
Limit[(x^2+y^2+z^2) Speriapsis[[2]]//.n → 1,y → ∞]
Limit[(x^2+y^2+z^2) Speriapsis[[3]]//.n → 1,z → ∞]
```

Out[35]= 0

Out[36]= 0

Out[37]= 0

In[38]:=
```
(*All zero. Success!*)
```