# SICs, ETFs and their Connections 

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#### Abstract

SIC-POVMs or Symmetric Informationally Complete-Positive Operator Valued Measures are special types of POVMs consisting of $d^{2}$ symmetrically placed projectors acting on the $d$-dimensional Hilbert space, hence making them informationally complete. Their significance is seen in Quantum Cryptography, Quantum State Tomography and various other areas of physics, mathematics and signal processing. Though there are numerous dimensions in which SICs have been found, convincingly suggesting they exist in every finite dimension, there is yet to be a proof of their existence. Moreover, it is difficult to find these SIC solutions. Thus, the search for SICs in higher dimensions and the search for a proof of existence are topics of extensive and dynamic research in the field.

In this work we provide the background for getting familiar with SIC-POVMs, focusing on the role of the Weyl-Heisenberg Group. We explore the connection between SIC-POVMs in lower dimensions to those in certain higher dimensions, creating a pathway between them using Equiangular Tight Frames (ETFs), a topic crucial to this thesis. We create new ETFs from existing SIC-POVMs, the final goal being the construction of a SIC-POVM in a higher dimension. An open question concerning ETFs that can be embedded in the higher dimensional SICPOVM is settled.


Keywords: SIC-POVMs, ETFs, Weyl-Heisenberg Group, Clifford Group, Naimark Theorem

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## Chapter 1

## Introduction

Quantum Information Theory has taken an important position in Information Science after the vast advances in experimental physics, most notably in Quantum Optics. As quantum information science deals with the information contained in a quantum system, it is necessary to study quantum measurements. A general measurement in quantum mechanics can be described by a set of positive operators called a Positive Operator-Valued Measure or a POVM, which act on the quantum state in order to extract information from it. For physical quantum systems, this action of the operators on the quantum state corresponds to an action taken during an experiment. The most well known example of a POVM is given by a set of operators $P_{i}=|i\rangle\langle i|$, where the vectors $|i\rangle$ form an orthonormal basis in dimension $d$. The operators satisfy the relations

$$
\begin{gathered}
\sum_{i=0}^{d-1} P_{i}=\mathbb{1} \\
P_{i} P_{j}=\delta_{i j} P_{i}
\end{gathered}
$$

In this thesis, we will focus on another special class of POVMs, called the Symmetric Informationally Complete-POVMs or the SIC ${ }^{1}-\mathrm{POVMs}$, which will be defined below.

In general, an informationally complete POVM consists of $d^{2}$ operators using which we can completely determine the quantum state by repeated measurements on a large number of identically prepared systems. This process of determining the quantum state is known as quantum state tomography. An informationally complete-POVM consisting of a symmetric set of operators is known as a SICPOVM. It is in a precise sense theoretically optimal for the purpose of quantum

[^1]state tomography [1]. SIC-POVMs also come into the picture in quantum cryptography [2] and in a branch of mathematics known as frame theory, as maximal equiangular lines. SIC-POVMs have also found usage in Number Theory [3], with their structure offering insights into many unanswered questions.

SIC-POVMs, generally known as SICs, can be given by a set of $d^{2}$ unit vectors $\left\{\left|\psi_{I}\right\rangle\right\}_{I=0}^{d^{2}}$, which obey

$$
\begin{gather*}
\frac{1}{d} \sum_{i=1}^{d^{2}}\left|\psi_{I}\right\rangle\left\langle\psi_{I}\right|=\mathbb{1}  \tag{1.1}\\
\left|\left\langle\psi_{I} \mid \psi_{J}\right\rangle\right|^{2}=\frac{1}{d+1} \text { where } I \neq J \tag{1.2}
\end{gather*}
$$

with the elements of the POVM given by $\left|\psi_{I}\right\rangle\left\langle\psi_{I}\right| / d$. The vectors $\left\{\left|\psi_{I}\right\rangle\right\}_{I=0}^{d^{2}}$ are known as SIC vectors. Even with its simple definition, as we will also see further in the text, it is extremely difficult to find these SICs in higher dimensions. Indeed, a pressing problem in the study of SICs is to prove their existence in every finite dimension. With these challenges and its wide range of applications, it is highly appealing to study SICs and a proof of existence or a technique for the construction of a SIC in any dimension would be highly beneficial.

Our current knowledge of SICs derives from the exact and numerical solutions found in numerous dimensions. As of now, exact solutions have been published in dimensions up to 21 [4] and a few others with the highest being 323 [5]. Numerical solutions have been published in dimensions up to 121 by Scott (2017) [6], up to 151 by Fuchs et al. (2017) [7] using the code given by Scott, and sporadically up to 2208. Many more exact and numerical solutions have been found in an unpublished work by Grassl [8]. Even though a mathematical proof is yet to be found, because of the large number of dimensions SICs have been found in till now, it is widely believed that they exist in every dimension.

Many aspects of quantum information theory are amalgamated with classical information theory in numerous applications in communications. One such example is the Heisenberg Group, which originated in quantum mechanics and now has usage in many engineering applications including signal processing [9], Fourier analysis [10] and radar and communications [9][11]. It is also of fundamental significance in our study of SIC-POVMs and hence we will study this group in detail. There are various versions of the group pertaining to different applications, with the one we use in this work called the Weyl-Heisenberg Group as a distinction. An important related group which comes into the picture is the

Clifford group, which we will use in our discussion to study the symmetries of SIC-POVMs. The Clifford Group has applications in Quantum Computing, more specifically in Error-Correcting Codes.

In this thesis, we focus on the connection between SICs in lower dimensions and those in higher dimensions, specifically on the link between the dimensions $d$ and $d(d-2)$. This special connection between the two dimensions comes from the part Number Theory plays in the construction of SICs [3]. The aim of looking at this particular case is to find a way to construct a SIC in dimension $d(d-2)$ given a SIC in dimension $d$. If such a formulation is found, SICs can be constructed recursively in infinitely many dimensions, the only limiting factor being the computational power available to us. The ultimate goal of this approach would be to solve the existence problem by finding connections between SICs in other ways as well and obtaining a method of constructing a SIC in any dimension. Given the speed by which solutions in higher dimensions are being discovered, this highly optimistic goal could be realized sooner than expected.

In Chapter 2 we will focus on gaining some background knowledge for the study of SICs. We begin the chapter by giving a short introduction to Group Theory and listing some frequently used terms. We then talk about modular arithmetic and the Chinese Remainder Theorem which gives us a recipe to deal with higher dimensional objects by splitting them into lower dimensional factors. The Chinese Remainder Theorem is essential when connecting SICs from a lower dimension to SICs in a higher dimension. We then discuss the WeylHeisenberg Group in detail and its representation pertaining to our work. The Weyl-Heisenberg Group takes a central role in the study of SICs, its importance illustrated by the fact that all but one solution of SICs that we have found till date are orbits under the Weyl-Heisenberg group. In fact, the odd SIC forms an orbit under a different version of the Heisenberg group. Lastly, we talk about the Clifford group and look at its construction in detail.

In Chapter 3 we give the definition of SIC-POVMs and give some solutions in lower dimensions, mainly in dimensions 2 and 3 . These solutions are much simpler compared to the exact solutions in higher dimensions and hence we can state the fiducials and discuss their structure. We also talk about equiangular tight frames (ETFs), which are symmetric POVMs, consisting of $d \leq N \leq d^{2}$ vectors in dimension $d$. We observe that a SIC is an equiangular tight frame consisting of $d^{2}$ vectors and is hence a maximal ETF. The property of a SIC being an ETF will later help us to construct new ETFs from a given SIC using a method known
as the Naimark extension theorem. We also discuss $t$-designs here, a structure which allows us to average over the Hilbert space easily by averaging over only a finite number of vectors. We see that a SIC is a 2-design and look at some properties regarding this that will come in use later.

In Chapter 4, we talk about the Number Theory aspect of SICs and how number fields play a part in describing these solutions. We make the connection between dimensions $d$ and $d(d-2)$ clear in relation to the number fields pertaining to their solutions. We also look at other empirical observations linking the two dimensions and put forward ways to utilize these connections for furthering our construction of SICs.

In Chapter 5, we first talk about the symmetries a SIC is found to have. Having the knowledge about the symmetries of an object helps in understanding its structure. In this case, we will talk about how various symmetries are linked to different dimensions. We then look at special dimensions of the form $d=3 k$. The SICs in these dimensions are known to have certain properties which we can use to our advantage in further chapters. We look at the reduced density matrices in dimension 3 of the quantum state created using these SICs and study their structure.

We then dive into Chapter 6, where we will use our knowledge obtained in the previous chapters to try to construct a SIC in dimension $d(d-2)$. We explain the Naimark extension theorem in detail and give the procedure for the construction of new ETFs from SICs using a construction that Renes et al. (2004) [12] gave in one of the two pioneering publications on SICs. (The other, independent, pioneering publication was Zauner's PhD thesis [13]). For this chapter, the representations of the Weyl-Heisenberg Group are a key factor and we will investigate in detail how the group works differently in even and odd dimensions.

In this work, the computational calculations are done using Wolfram Mathematica 12.0 Student Edition and some useful code snippets are presented in Appendix A. We refrain from providing longer codes to avoid making the work cumbersome.

## Chapter 2

## Background

To study the basics of SIC-POVMs, we first need to get some mathematical background for its fundamental concepts. As one of the central themes in the study of SICs is the Weyl-Heisenberg Group, we start with a discussion of Groups before moving on to modular arithmetic and the Chinese Remainder Theorem, which are essential to understand the Weyl-Heisenberg group. We then talk about the Clifford group which also has an important role in this work and as we will explain below, is the normalizer of the Weyl-Heisenberg group.

### 2.1 Group Theory

Group Theory deals with the study of groups, a mathematical object first introduced in the early nineteenth century for finding the solutions of higher degree polynomial equations. Over the years there have been numerous developments of the theory, and as it stands now it is essential to many applications in physics. Group representation theory, in particular, is widely used in Quantum Mechanics, as was first realized by Weyl [14]. It is important for the SIC problem as well, and although we will not enter deeply into the aspect, it turns out that the original use of group theory by Galois is also highly important for SICs [15]. Though the subject area is vast and diverse, we will give some basic definitions and focus mainly on topics pertaining to our study of SICs.

A group $G$ is defined as a set of elements, which together with a binary operation $\circ$, follow the group axioms -

Closure: The group elements are closed under the binary operation o, i.e., for all elements $a, b \in G$

$$
a \circ b=c
$$

such that $c$ is an element of $G$ as well.

Associativity: The operation $\circ$ is associative, i.e., for all $a, b, c \in G$

$$
(a \circ b) \circ c=a \circ(b \circ c)
$$

Identity: There exists a unique element $e$ in $G$ such that for any element $a \in G$,

$$
a \circ e=a=e \circ a
$$

$e$ is called the identity element of the group $G$.
Invertibility: For all elements $a \in G$, there exists an element $a^{\prime}$ belonging to $G$ which acts as its inverse

$$
a \circ a^{\prime}=e=a^{\prime} \circ a
$$

Given a set of elements and a corresponding binary operation, we can easily check if the elements form a group by verifying that they follow the group axioms above. An example of a group is the set of all real numbers $\mathbb{R}$, which forms a group under addition. The identity element $e$ for the group is 0 and the inverse of an element $g$ is given by $-g$. We can call this group as $(\mathbb{R},+)$.

A subgroup, as is evident from the name, is a subset of the original set of elements of $G$, which forms a group under the operation $\circ$ as well. For a subset $H$ to be a subgroup of $G$, it must follow the group axioms. As an example, a subgroup of $(\mathbb{R},+)$ is the set of all integers $\mathbb{Z}$ forming a group under addition, $(\mathbb{Z},+)$.

With this we can define the concept of a normalizer of a group. Given a group $G$ and its subgroup $H$, let $N_{G}(H)$ be the set of all elements $n \in G$ such that

$$
n H n^{-1}=H
$$

Then, the group $N_{G}(H)$ is called the normalizer of $H$.
A group $(G, \circ)$ is called abelian if $a \circ b=b \circ a$ for all elements $a, b \in G$. The group operation o thus becomes commutative. The group created by the set of real numbers $\mathbb{R}$ under addition is an abelian group. An example of a non-abelian group is the rotation group $S O(3)$, which consists of orthogonal 3-dimensional matrices with determinant 1 as group elements, under the operation of matrix
multiplication. As we will see later, the Weyl-Heisenberg group is a non-abelian group as well.

There are some special notations given to specific mappings in group theory. A function which maps the group $(G, \circ)$ to $(H,$.$) is known as a group homomor-$ phism if it preserves the group operations.

$$
\begin{aligned}
& f: G \rightarrow H \\
& f\left(g_{1} \circ g_{2}\right)=f\left(g_{1}\right) \cdot f\left(g_{2}\right)
\end{aligned}
$$

for elements $g_{1}$ and $g_{2}$ in group $G$. If a homomorphism is bijective, i.e., a one-to-one and onto map, it is known as a group isomorphism. The two groups $G$ and $H$ are then isomorphic to each other. As there is a unique correspondence between the elements $g_{k}$ and $h_{k}$, the groups $G$ and $H$ are essentially the same, in the sense that the element $h_{k}$ acts as a label for $g_{k}$.

### 2.1.1 Group Representation

Given a group $G$, the group elements $g$ can be represented as matrices $\Gamma(g)$ such that

$$
\begin{equation*}
\Gamma\left(g_{1}\right) \cdot \Gamma\left(g_{2}\right)=\Gamma\left(g_{1} \circ g_{2}\right) \tag{2.1}
\end{equation*}
$$

Here, $(\cdot)$ represents matrix multiplication and $(\circ)$ is the binary operation corresponding to the group $G$. The representations are non-unique and hence we can have different matrix representations for $G$ with the representations $\Gamma(g)$ and $\tilde{\Gamma}(g)$ being equivalent if there exists a transformation such that

$$
\tilde{\Gamma}(g)=S^{-1} \Gamma(g) S
$$

An illustration of a matrix representation of the additive group of real numbers $(\mathbb{R},+)$ can be given by the group of lower triangular 2-dimensional matrices with matrix multiplication as their operation.

$$
\Gamma(u)=\left(\begin{array}{ll}
1 & 0  \tag{2.2}\\
u & 1
\end{array}\right)
$$

such that

$$
\Gamma(u) \cdot \Gamma(v)=\left(\begin{array}{cc}
1 & 0  \tag{2.3}\\
u+v & 1
\end{array}\right)
$$

and

$$
e=\Gamma(0)=\left(\begin{array}{ll}
1 & 0  \tag{2.4}\\
0 & 1
\end{array}\right)
$$

We will mainly deal with the matrix representations of the two groups, the WeylHeisenberg group and the Clifford group, in this work. In the case of the WeylHeisenberg group, the different representations a particular group can take is of significance and will come in use later.

A representation $\Gamma(g)$ is said to be reducible if there exists an equivalent representation of $\Gamma(g)$ which takes the form

$$
\left(\begin{array}{c:c}
D_{1} & A_{1} \\
\hdashline \mathbf{0} & D_{2}
\end{array}\right)
$$

where the elements of the matrix are themselves matrices of varying sizes. A completely reducible representation is given by a matrix

$$
\left(\begin{array}{c:c}
D_{1} & \mathbf{0} \\
\hdashline \mathbf{0} & D_{2}
\end{array}\right)
$$

Here, the diagonal elements $D_{1}$ and $D_{2}$ are square matrices of varying sizes and the rest of the elements are zero matrices. Such a matrix is called a block diagonal matrix. In general, a block diagonal matrix can have many blocks on the diagonal depending on the dimension of the matrix. For a finite group (a group containing a finite number of elements), which are the kind of groups we will be working with in this thesis, the reducible and completely reducible representations are equivalent. If a matrix representation cannot take this form, it is called an irreducible representation.

### 2.1.2 Special Linear Group

The special linear group $S L(n, F)$ is the set of $n$-dimensional matrices with determinant 1, over the ring $F$ with matrix multiplication as the group operation. A ring is an abelian group under addition, and is closed under multiplication. An example of a ring is the set of integers $\mathbb{Z}$. If a ring is also closed under division, it is called a field. $\mathbb{R}$ and $\mathbb{C}$, the sets of real numbers and complex numbers respectively, are hence fields. For example, the group $S L(2, \mathbb{Z})$ is the set of 2 -dimensional matrices

$$
\left(\begin{array}{ll}
\alpha & \beta \\
\gamma & \delta
\end{array}\right)
$$

such that

$$
\alpha, \beta, \gamma, \delta \in \mathbb{Z} \quad \text { and } \quad \alpha \delta-\beta \gamma=1
$$

Similarly, we can encounter other special linear groups $S L(2, \mathbb{R}), S L(2, \mathbb{C}), S L\left(2, \mathbb{Z}_{n}\right)$, etc. with a similar construction. The group $S L\left(2, \mathbb{Z}_{n}\right)$ will be of importance to us when we define the Clifford group, the normalizer of the Weyl-Heisenberg group. To understand the set $\mathbb{Z}_{n}$, which is the set of all integers taken modulo $n$, we will discuss the concept of modulo and modular arithmetic at length below. As we will later see, modular arithmetic will be of great importance to us in the study of the Weyl-Heisenberg Group as well.

### 2.2 Modular Arithmetic

The concept of modulo can easily be understood by the most probable cause of its conception: keeping track of astronomical events. As a simple example, the time of day resets every 24 hours. So any calculation done can be reset to 0 when the counter reaches 24 . This arithmetic is said to be modulo 24 . As astronomical events are mostly cyclic, the position of a body being equivalent after a complete revolution, the calendric calculations can be done modulo the period of revolution of the astronomical body. As a consequence, modular arithmetic was being used in India to do such calculations, possibly as early as 700 BC [16]. It was also being used by the Ancient Greeks, Arabs and Chinese, evidence suggesting independent creation.

In modular arithmetic, two integers $a$ and $b$ are called equivalent with respect to a positive integer $n$ if $n$ is a factor of $a-b$. The arithmetic is then modulo $n$ and the relation is given by

$$
a \equiv b(\bmod n)
$$

Hence, it follows that any number $a$ is equivalent to $\ldots a-n, a+n, a+2 n, \ldots$ $(\bmod n)$. We are dividing the set of integers into $n$ different equivalence classes with $n$ called the modulus. An easy way to think of it is to say that a number is equivalent to the remainder we obtain while dividing it by the modulus, keeping the quotient an integer. An everyday application of modular arithmetic is the classification of integers into odd or even. An integer is defined as even if it is 0 $\bmod 2$ and odd if it is $1 \bmod 2$.

Being an equivalence relation, modulo follows the properties

$$
\text { - } a \equiv a(\bmod n) \quad \text { (reflexivity) }
$$

- $a \equiv b(\bmod n) \Rightarrow b \equiv a(\bmod n)$
- $a \equiv b(\bmod n)$ and $b \equiv c(\bmod n) \Rightarrow a \equiv c(\bmod n)$
(symmetry)
(transitivity)

The set of integers modulo $n$ is denoted by $\mathbb{Z}_{n}$. This set is a ring as it is closed under the operations addition and multiplication. We should get familiar with how addition and multiplication operates in modular arithmetic. Some properties that integers modulo $n$ follow are -

- $a-b n=a(\bmod n), b \in \mathbb{Z}$
- $a(\bmod n)+b(\bmod n)=(a+b)(\bmod n)$
- if $a=b(\bmod n)$ then $a+k=b+k(\bmod n)$ for any $k \in \mathbb{Z}$
- $a(\bmod n) \cdot b(\bmod n)=(a . b)(\bmod n)$
- if $a=b(\bmod n)$ then $a k=b k(\bmod n)$ for any $k \in \mathbb{Z}$

After talking about the operations addition and multiplication, we would like to know how division comes into the picture in modular arithmetic. Let us start with two numbers $a$ and $b$ which are co-prime, i.e. they have no common factors other than 1 . Co-prime numbers are denoted by $(a, b)=1$. The inverse of a number $a$ modulo $n$ has an existence only if the numbers $a$ and $n$ are co-prime. The inverse then is given by $a^{-1}$ where $a a^{-1}=1(\bmod n)$. For example, $3^{-1}(\bmod 5)=2$ as $3 \cdot 2(\bmod 5)=6(\bmod 5)=1(\bmod 5)$. This puts forward the notion that for division to be a part of the operations possible to perform, all non-zero integers should have an inverse modulo $n$. Thus, the set $\mathbb{Z}_{n}$ is a field only if $n$ is prime and otherwise it is only a ring.

An important concept that will come in use later is the quadratic residue. An integer $x$ is called a quadratic residue modulo $n$ if there exists an integer $y$ such that $y^{2}=x(\bmod n)$. As an example, -1 is a quadratic residue when taken modulo 5 or modulo 13 as

$$
\begin{aligned}
3^{2}(\bmod 5) & =9(\bmod 5)=-1(\bmod 5) \\
5^{2}(\bmod 13) & =25(\bmod 13)=-1(\bmod 13)
\end{aligned}
$$

However, for modulo $3,7,9,11$ there is no such number whose square is equivalent to -1 . The concept of a quadratic residue is analogous to classifying real numbers into positive or negative. One particular definition states that a real number is positive if it is found to be the square of another real number, i.e., it is a quadratic residue for real numbers. Otherwise, it is defined as a negative real number.

### 2.3 Chinese Remainder Theorem

Now that we have understood modular arithmetic and know its properties, we should ask ourselves how to solve equations which are given modulo different numbers. Even though the calculations would be easy for smaller numbers, modular arithmetic becomes extremely difficult to deal with if the moduli we're dealing with is larger. There are many algorithms [16] which tell us how to attack this problem, one of the oldest being the Chinese Remainder Theorem.

From one point of view, the Chinese Remainder Theorem is a method of solving problems involving many equations modulo a large range of numbers. The theorem is stated as follows -

A system of linear equations

$$
\begin{gathered}
x \equiv b_{1}\left(\bmod n_{1}\right) \\
x \equiv b_{2}\left(\bmod n_{2}\right) \\
\vdots \\
x \equiv b_{3}\left(\bmod n_{r}\right)
\end{gathered}
$$

where $n_{1}, n_{2}, \ldots n_{r}$ are pairwise co-prime positive integers, has a unique solution modulo $n_{1} n_{2} \ldots n_{r}$.

As an example let's take the equations

$$
\begin{aligned}
& x=2(\bmod 3) \\
& x=1(\bmod 5)
\end{aligned}
$$

$x$ can be $11,26,41$ etc. but the solutions are equivalent modulo 15. Although algorithms to solve this system of equations were being used in China and India to solve problems in astronomy around the $3^{\text {rd }}$ century [16], Qin Jiushao gave the complete proof of the Chinese Remainder Theorem in 1247 [17].

We can go in the reverse direction and observe that if we have a problem modulo a large number, we can split it into its pairwise co-prime factors and solve the problem modulo smaller numbers. Then, there is a unique way to reconstruct the solution modulo the larger number. The quality to break a larger problem into smaller blocks to solve individually is highly desirable as solving problems
modulo smaller numbers is generally much easier and hence reduces calculation time as well as complexity issues.

For $d=d_{1} d_{2}$ where $\left(d_{1}, d_{2}\right)=1$,

$$
\begin{align*}
& r=r_{1}\left(\bmod d_{1}\right)  \tag{2.5}\\
& r=r_{2}\left(\bmod d_{2}\right) \tag{2.6}
\end{align*}
$$

where $r$ is computed modulo $d$, the unique solution is given by[18]

$$
\begin{equation*}
r=r_{1} d_{2} d_{2}^{-1}+r_{2} d_{1} d_{1}^{-1} \tag{2.7}
\end{equation*}
$$

Here $d_{1}^{-1}$ is calculated modulo $d_{2}$ and $d_{2}^{-1}$ is calculated modulo $d_{1}$.
To prove this is the solution one can assume (3) and then easily see

$$
\begin{equation*}
r\left(\bmod d_{1}\right)=\left(r_{1} d_{2} d_{2}^{-1}+r_{2} d_{1} d_{1}^{-1}\right)\left(\bmod d_{1}\right)=r_{1} \tag{2.8}
\end{equation*}
$$

Similarly,

$$
\begin{equation*}
r\left(\bmod d_{2}\right)=r_{2} \tag{2.9}
\end{equation*}
$$

Using the symmetry of modular arithmetic, we see that (4) and (5) are equivalent to (1) and (2). Hence, (3) is a solution. However, we still need to prove that it is the only solution.

Let there be two solutions $R_{1}$ and $R_{2}$ modulo $d$. Then,

$$
\begin{aligned}
R_{1} & =r_{1}\left(\bmod d_{1}\right) \\
R_{2} & =r_{1}\left(\bmod d_{1}\right) \\
R_{1}-R_{2} & =0\left(\bmod d_{1}\right)
\end{aligned}
$$

So, $R_{1}-R_{2}$ is a multiple of $d_{1}$. Similarly, $R_{1}-R_{2}$ is a multiple of $d_{2}$. As $d_{1}$ and $d_{2}$ are co-prime, $d_{1} d_{2}$ is a factor of $R_{1}-R_{2}$.

$$
\begin{aligned}
R_{1}-R_{2} & =0 \quad(\bmod d) \\
R_{1} & =R_{2}(\bmod d)
\end{aligned}
$$

So, we see that equation (3) is a unique solution modulo $d$.

We now have a brief but informative background on groups, modular arithmetic and the Chinese Remainder Theorem. These are essential to understanding the
next and one of the most important concepts in our study, the Weyl-Heisenberg Group.

### 2.4 Weyl-Heisenberg Group

The Weyl-Heisenberg Group is central to our work and hence we shall look at it in detail. As mentioned in Section 2.1, a group is a set of elements which together with a binary operation follow the group axioms. The Weyl-Heisenberg Group is defined by the generators of its group, $\omega, X$ and $Z$ which follow the conditions

$$
\begin{gather*}
\omega^{d}=X^{d}=Z^{d}=\mathbb{1}  \tag{2.10}\\
Z X=\omega X Z \tag{2.11}
\end{gather*}
$$

and $\omega$ commutes with everything. The group is then created by going through all combinations of these generators, i.e., the generators are used as "letters" to create a "word" which is an element in the group. Two words are considered the same if they can be transformed into each other using the given relations. The general group element can then be brought to the form

$$
\begin{equation*}
\omega^{l} X^{m} Z^{n} \tag{2.12}
\end{equation*}
$$

We thus have a total of $d^{3}$ elements in the group.

We will now look at the group representations for the Weyl-Heisenberg group. For $\omega$ to commute with every element in an irreducible representation, it must be proportional to the unit matrix. We also insist that $\omega^{d}=\mathbb{1}$, hence the factor is a $d^{t h}$ root of unity. For a faithful representation we insist that $\omega=\left(e^{\frac{2 \pi i}{d}}\right)^{n} \neq 1$ for any $n<d$ and so it is a primitive root of unity, i.e., a number which is not the root of unity for any integer less than $d$. We now make a choice to put $n=1$. And so,

$$
\begin{equation*}
\omega=e^{\frac{2 \pi i}{d} \mathbb{1}} \tag{2.13}
\end{equation*}
$$

We now introduce the eigenvalue basis for $Z$ and let

$$
\begin{equation*}
Z\left|e_{0}\right\rangle=\left|e_{0}\right\rangle \tag{2.14}
\end{equation*}
$$

Then, according to 2.11

$$
\begin{equation*}
Z X\left|e_{0}\right\rangle=\omega X Z\left|e_{0}\right\rangle=\omega X\left|e_{0}\right\rangle \tag{2.15}
\end{equation*}
$$

We define $\left|e_{1}\right\rangle$ such that

$$
\begin{align*}
X\left|e_{0}\right\rangle & =\left|e_{1}\right\rangle  \tag{2.16}\\
\Rightarrow Z\left|e_{1}\right\rangle & =\omega\left|e_{1}\right\rangle \tag{2.17}
\end{align*}
$$

Similarly, defining $\left|e_{2}\right\rangle$

$$
\begin{align*}
X\left|e_{1}\right\rangle & =\left|e_{2}\right\rangle  \tag{2.18}\\
\Rightarrow Z\left|e_{2}\right\rangle & =\omega^{2}\left|e_{2}\right\rangle \tag{2.19}
\end{align*}
$$

Doing this repeatedly gives

$$
\begin{align*}
X\left|e_{r}\right\rangle & =\left|e_{r+1}\right\rangle \&  \tag{2.20}\\
Z\left|e_{r}\right\rangle & =\omega^{r}\left|e_{r}\right\rangle \tag{2.21}
\end{align*}
$$

where the indices are modulo $d$. We see that no degenerate eigenvalues occur in $Z$. Also, as $Z$ is of order $d$, its eigenvalues have to be powers of $\omega$ which is consistent with our outcome. The above equations give us [14]

$$
\begin{equation*}
X=\sum_{r=0}^{d-1}|r+1\rangle\langle r| \tag{2.22}
\end{equation*}
$$

and

$$
\begin{equation*}
Z=\sum_{r=0}^{d-1} \omega^{r}|r\rangle\langle r| \tag{2.23}
\end{equation*}
$$

and in the matrix form

$$
\begin{align*}
X & =\left(\begin{array}{ccccc}
0 & 0 & \cdots & 0 & 1 \\
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0
\end{array}\right)_{d \times d}  \tag{2.24}\\
Z & =\left(\begin{array}{cccc}
1 & 0 & \cdots & 0 \\
0 & \omega & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \omega^{d-1}
\end{array}\right)_{d \times d} \tag{2.25}
\end{align*}
$$

We now have a unitary irreducible representation of the group which is "essentially" unique as we made a choice for $\omega$ and used the eigenvector basis for $Z$.

Looking at the matrix forms, it is also quite simple to create these generators and hence the group elements in any dimension. $X$ turns out to be a permutation matrix, i.e., a matrix with a single 1 in each row and column and 0's elsewhere. We only need to calculate $\omega$ and its powers in the particular dimension $d$ for creating $Z$ and henceforth the group elements.

We now define the Displacement Operators as [19],

$$
\begin{equation*}
D_{i, j}=\tau^{i j} X^{i} Z^{j} \tag{2.26}
\end{equation*}
$$

where

$$
\begin{equation*}
\tau=-e^{\frac{i \pi}{d}} \tag{2.27}
\end{equation*}
$$

We note that here we are not using $\omega$, which we had defined earlier, in the definition of the operators. It might look peculiar to introduce $\tau$ instead of using $\omega$, since $\tau$ is the $2 d^{\text {th }}$ root of unity when $d$ is even and a power of $\omega$ when $d$ is odd, we will see later that it was introduced by Appleby [19] for convenience. Clearly, $\tau$ has a relation to $\omega$ given by

$$
\begin{equation*}
\tau^{2}=\omega \tag{2.28}
\end{equation*}
$$

This difference in odd and even dimensions led Appleby to introduce the notation [19],

$$
\bar{d}= \begin{cases}d & \mathrm{~d} \text { is odd }  \tag{2.29}\\ 2 d & \mathrm{~d} \text { is even }\end{cases}
$$

This notation will come into use in several places ahead, specially in the topic of Clifford Groups.

Let us now define $(i, j)$ as a vector $\mathbf{p}$.

$$
\begin{equation*}
\mathbf{p}=\binom{i}{j} \tag{2.30}
\end{equation*}
$$

to rewrite the displacement operator as $D_{\mathbf{p}}$. As $i$ and $j$ take values from 0 to $d-1$, we get $d^{2}$ displacement operators. These operators are a selection made out of the $d^{3}$ elements of the Weyl-Heisenberg group by ignoring the phase factors.

The displacement vectors have the following properties [19]

$$
\begin{gather*}
D_{\mathbf{p}}^{\dagger}=D_{-\mathbf{p}}  \tag{2.31}\\
D_{\mathbf{p}} D_{\mathbf{q}}=\tau^{\langle\mathbf{p}, \mathbf{q}\rangle} D_{\mathbf{p}+\mathbf{q}} \tag{2.32}
\end{gather*}
$$

$$
\begin{equation*}
\langle\mathbf{p}, \mathbf{q}\rangle=p_{2} q_{1}-p_{1} q_{2} \tag{2.33}
\end{equation*}
$$

Here, $\langle\mathbf{p}, \mathbf{q}\rangle$ is called the symplectic form. We see that the introduction of $\tau$ has made eq 2.32 take a nice form.

As we can see from equations 2.22 and 2.23 , the Displacement Operators obey

$$
\begin{equation*}
\operatorname{Tr} D_{\mathbf{p}}=0, \text { for } \mathbf{p} \neq(0,0) \tag{2.34}
\end{equation*}
$$

It follows that the trace inner product of the displacement operators is [20]

$$
\begin{equation*}
\operatorname{Tr} D_{\mathbf{p}}^{\dagger} D_{\mathbf{q}}=0, \text { for } \mathbf{p} \neq \mathbf{q} \tag{2.35}
\end{equation*}
$$

These $d^{2}$ operators are hence orthogonal to each other and as the set of all operators has dimension $d^{2}$, they form a basis in this space. As these operators are unitary, they form a Unitary Operator Basis [21]. This is an important property of the group and one we will use later.

An important instance of the group can be found looking at the operators $X$ and $Z$ in $d=2$. We get

$$
\begin{gathered}
D_{1,0}=X=\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right) \\
D_{0,1}=Z=\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right)
\end{gathered}
$$

which are easily recognizable as the Pauli matrices $\sigma_{x}$ and $\sigma_{z}$ respectively. Also,

$$
-D_{1,1}=\left(\begin{array}{cc}
0 & -i  \tag{2.36}\\
i & 0
\end{array}\right)=\sigma_{y}
$$

We again see the significance of introducing $\tau$ as for $d=2, \tau=-i$ and so we see the Pauli Matrices represented by the displacement operators. This connection with the Pauli matrices is the reason why the generators of the Weyl-Heisenberg group were named $X$ and $Z$ and why the group is sometimes also known as the Generalized Pauli Group.

### 2.5 Chinese Remainder Theorem for the WeylHeisenberg Group

We have now seen the Weyl-Heisenberg Group and its generators,

$$
\begin{gathered}
\omega=e^{\frac{2 \pi i}{d} \mathbb{1}} \\
X\left|e_{r}\right\rangle=\left|e_{r+1}\right\rangle \\
Z\left|e_{r}\right\rangle=\omega^{r}\left|e_{r}\right\rangle
\end{gathered}
$$

where the indices on the vectors are modulo $d$. Also, the generator matrices themselves are of order $d$. With this we can say that the Weyl-Heisenberg Group can be characterized by operations modulo $d$. Because of the direct dependence of the group on the dimension, it can be inferred that higher dimensions might bring more complexity issues. Recalling that the Chinese Remainder Theorem can be applied in situations concerning modulo expressions, we use it to split the displacement operator $D_{i, j}^{(d)}$ into a tensor product of $D_{i, j}$ 's in smaller dimensions.

Let $d=d_{1} d_{2}$ and $\left(d_{1}, d_{2}\right)=1$. Then, $d_{1}^{-1}$ is the inverse of $d_{1}$ modulo $d_{2}$ and similarly $d_{2}^{-1}$ is the inverse of $d_{2}$ modulo $d_{1}$. Then, [18]

$$
\begin{gathered}
\omega_{d}=e^{\frac{2 \pi i}{d}}=e^{\frac{2 \pi i}{d_{1} d_{2}} \cdot 1}=e^{\frac{2 \pi i}{d_{1} d_{2}}\left(d_{1} d_{1}^{-1}+d_{2} d_{2}^{-1}\right)}=e^{\frac{2 \pi i}{d_{1}} d_{2}^{-1}} e^{\frac{2 \pi i}{d_{2}} d_{1}^{-1}} \\
\omega_{d}=\omega_{d_{1}}^{d_{1}^{-1}} \omega_{d_{2}}^{d_{1}^{-1}}
\end{gathered}
$$

Looking first at the relation,

$$
\begin{align*}
Z_{d}\left|e_{r}\right\rangle_{d} & =\omega_{d}^{r}\left|e_{r}\right\rangle_{d}  \tag{2.37}\\
& =\left(\omega_{d_{1}}^{d_{2}^{-1}} \omega_{d_{2}}^{d_{1}^{-1}}\right)^{r}\left(\left|e_{r_{1}}\right\rangle_{d_{1}} \otimes\left|e_{r_{2}}\right\rangle_{d_{2}}\right)  \tag{2.38}\\
& =Z_{d_{1}}^{d_{2}^{-1}}\left|e_{r_{1}}\right\rangle_{d_{1}} \otimes Z_{d_{2}}^{d_{1}^{-1}}\left|e_{r_{2}}\right\rangle_{d_{2}} \tag{2.39}
\end{align*}
$$

Doing a similar calculation for $X$, we get the following relations

$$
\begin{gather*}
Z_{d}=Z_{d_{1}}^{d_{1}^{-1}} \otimes Z_{d_{2}}^{d_{1}^{-1}}  \tag{2.40}\\
X_{d}=X_{d_{1}} \otimes X_{d_{2}} \tag{2.41}
\end{gather*}
$$

Now, we can see that the Chinese remaindering on the Displacement Operators gives

$$
D_{i, j}^{(d)}=D_{i, d_{2}^{-1}{ }_{j}}^{\left(d_{1}\right)} \otimes D_{i, d_{1}^{-1}{ }_{j}}^{\left(d_{2}\right)}
$$

Or, if we have the matrices

$$
\begin{gather*}
H_{1}=\left(\begin{array}{cc}
1 & 0 \\
0 & d_{2}^{-1}
\end{array}\right) \quad H_{2}=\left(\begin{array}{cc}
1 & 0 \\
0 & d_{1}^{-1}
\end{array}\right)  \tag{2.42}\\
D_{\mathbf{p}}^{(d)}=D_{H_{1} \mathbf{p}}^{\left(d_{1}\right)} \otimes D_{H_{2} \mathbf{p}}^{\left(d_{2}\right)}
\end{gather*}
$$

If instead we had taken $\omega_{d_{1}}^{d_{2}^{-1}}$ as the new primitive root of unity in dimension $d_{1}$ and similarly $\omega_{d_{2}}^{d_{1}^{-1}}$ in $d_{2}$, the Chinese remaindering relation becomes much simpler. However, we prefer not to do this since we want eq 2.11 to hold in any dimension.

We can now have problems in a higher dimension which we can solve in the lower dimensional co-prime factors and uniquely rebuild a solution in the higher dimension. This not only simplifies the calculation but as we'll later see, Chinese remaindering of the Weyl-Heisenberg Group has a significance in connecting the SICs from a lower dimension to a higher dimension.

After discussing the Weyl-Heisenberg Group and its Chinese Remaindering, the next step is to study another important group to be used in this work, the Clifford Group. Before diving into this topic, we will give a little background on Anti-Unitary Operators, which are useful in the discussion of the Clifford Group.

### 2.6 Anti-Unitary Operators

An Anti-Unitary operator $A$ is defined as [22]

$$
\begin{equation*}
\langle A \phi \mid A \psi\rangle=\langle\phi \mid \psi\rangle^{*}=\langle\psi \mid \phi\rangle \tag{2.43}
\end{equation*}
$$

In quantum mechanics, anti-unitary operators are used to define the time-reversal symmetry. These operators preserve the value $|\langle\psi \mid \phi\rangle|^{2}$, and come in use in the study of SICs while investigating the symmetries in different dimensions. Just as a Unitary operator is linear

$$
\begin{equation*}
U(a|\phi\rangle+b|\psi\rangle)=a(U|\phi\rangle)+b(U|\psi\rangle), \quad a, b \in \mathbb{C}, \tag{2.44}
\end{equation*}
$$

anti-unitary operations are anti-linear, i.e.

$$
\begin{equation*}
A(a|\phi\rangle+b|\psi\rangle)=a^{*}(A|\phi\rangle)+b^{*}(A|\psi\rangle) \quad a, b \in \mathbb{C} \tag{2.45}
\end{equation*}
$$

For an anti-unitary operator $A, A^{2}$ is a unitary by definition.

$$
\begin{equation*}
\left\langle A^{2} \phi \mid A^{2} \psi\right\rangle=\langle A \psi \mid A \phi\rangle=\langle\phi \mid \psi\rangle \tag{2.46}
\end{equation*}
$$

Furthermore, any anti-unitary operator can be decomposed as $A=U K$ where $U$ is a unitary operator and $K$ is the operation of complex conjugation in the computational basis.

$$
\begin{gather*}
K|\psi\rangle=|\psi\rangle^{*}  \tag{2.47}\\
K^{-1}=K ; \quad K^{2}=1  \tag{2.48}\\
K U=U^{*} K \tag{2.49}
\end{gather*}
$$

With the basic properties of anti-unitary operators understood, we shall proceed to the second significant group that occurs in this work, the Clifford Group.

### 2.7 Clifford Group

The normalizer of the Weyl-Heisenberg Group within the unitary group is called the Clifford Group, denoted by $C(d)$ in dimension $d$. It is the set of all unitary operators such that

$$
\begin{equation*}
U D_{\mathbf{p}} U^{\dagger} \doteq D_{\mathbf{p}^{\prime}} \tag{2.50}
\end{equation*}
$$

where $\doteq$ means equivalent up to a phase factor.
For the construction of such unitaries, we look towards Lemma 2 in Appleby (2005) [19], where it is proved that for a matrix $F \in S L\left(2, \mathbb{Z}_{\bar{d}}\right)$,

$$
\begin{gather*}
F=\left(\begin{array}{cc}
\alpha & \beta \\
\gamma & \delta
\end{array}\right)  \tag{2.51}\\
\alpha, \beta, \gamma, \delta \in \mathbb{Z}_{\bar{d}} \quad \text { and } \quad \alpha \delta-\beta \gamma=1(\bmod \bar{d}) \tag{2.52}
\end{gather*}
$$

we can construct a unitary $V_{F}$ using the matrix elements of $F$ as

$$
\begin{equation*}
V_{F}=\frac{1}{\sqrt{d}} \sum_{r, s=0}^{d-1} \tau^{\beta^{-1}\left(\alpha s^{2}-2 r s+\delta r^{2}\right)}\left|e_{r}\right\rangle\left\langle e_{s}\right| \tag{2.53}
\end{equation*}
$$

Here, $\bar{d}$ follows the notation defined in eq. 2.29. These unitaries then act on the displacement operators following the relation

$$
\begin{equation*}
V_{F} D_{\mathbf{p}} V_{F}^{\dagger}=D_{F \mathbf{p}} \tag{2.54}
\end{equation*}
$$

Clearly, this construction of a unitary works only if $\beta$ has an inverse $\bmod \bar{d}$, i.e., if $\beta \neq 0$ and $(\beta, \bar{d})=1$. If either of the two conditions are not satisfied, we are also given a roundabout. We can split the matrix $F$ in a way

$$
\begin{gather*}
F=F_{1} F_{2}  \tag{2.55}\\
F_{1}=\left(\begin{array}{cc}
0 & -1 \\
1 & x
\end{array}\right)  \tag{2.56}\\
F_{2}=\left(\begin{array}{cc}
\gamma+x \alpha & \delta+x \beta \\
-\alpha & -\beta
\end{array}\right) \tag{2.57}
\end{gather*}
$$

such that $(\delta+x \beta, \bar{d})=1$. Then the construction becomes

$$
\begin{equation*}
V_{F}=V_{F_{1}} V_{F_{2}} \tag{2.58}
\end{equation*}
$$

If $\beta=0, \delta$ is non-zero to fulfill the condition $\alpha \delta-\beta \gamma=1(\bmod \bar{d})$ and hence $x$ can be taken as 0 . This particular case of $\beta=0$ gives rise to a simple nature of the unitary formed. Such a matrix $F$, acting on the $Z$ operator, transforms it into a power of $Z$. As $Z$ is a diagonal matrix with eigenvalues $1, \omega, \omega^{2}, \ldots \omega^{d-1}$, a power of $Z$ is just a permutation transformation reordering the eigenvalues in the diagonal matrix. Hence, a unitary created by the matrix $F$ having $\beta=0$ is a permutation matrix.

$$
\begin{align*}
& F=\left(\begin{array}{ll}
\alpha & 0 \\
\beta & \gamma
\end{array}\right)  \tag{2.59}\\
& F \mathbf{p}=\left(\begin{array}{ll}
\alpha & 0 \\
\beta & \gamma
\end{array}\right)\binom{0}{j}=\binom{0}{\gamma j}  \tag{2.60}\\
& \left.U_{\left(\begin{array}{ll}
\alpha & 0 \\
\beta & \gamma
\end{array}\right)}^{D}\binom{0}{j}^{U^{\dagger}} \begin{array}{ll}
\alpha & 0 \\
\beta & \gamma
\end{array}\right)=D\binom{0}{\gamma j} \tag{2.61}
\end{align*}
$$

Chinese Remaindering can be applied to the Clifford Group as well, following the same logic as in the case of the Weyl-Heisenberg Group. Using matrices $H_{1}$ and $\mathrm{H}_{2}$ from 2.42, we find that [18]

$$
\begin{equation*}
U_{F}^{(d)}=U_{H_{1} F H_{1}^{-1}}^{\left(d_{1}\right)} \otimes U_{H_{2} F H_{2}^{-1}}^{\left(d_{2}\right)} \tag{2.62}
\end{equation*}
$$

### 2.7.1 Extended Clifford Group

We can now define the Extended Clifford Group, $E C(d)$, which adds the set of anti-unitary operators $A$ 's to the unitaries of the Clifford Group, such that the
following holds.

$$
\begin{equation*}
A D_{\mathbf{p}} A^{\dagger} \doteq D_{\mathbf{p}^{\prime}} \tag{2.63}
\end{equation*}
$$

While a unitary operator in $C(d)$ is characterized by $S L\left(2, Z_{\bar{d}}\right)$, the anti-unitaries are characterized by the Extended Special Linear group, $E S L\left(2, Z_{\bar{d}}\right.$, which modifies the former such that

$$
\begin{equation*}
\alpha \delta-\beta \gamma= \pm 1(\bmod \bar{d}) \tag{2.64}
\end{equation*}
$$

As $A=U K$,

$$
\begin{gather*}
\left(\begin{array}{cc}
\alpha & \beta \\
\gamma & \delta
\end{array}\right)_{(\operatorname{det}-1)}=\left(\begin{array}{ll}
\alpha & -\beta \\
\gamma & -\delta
\end{array}\right)_{(\operatorname{det} 1)}\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right)_{(\operatorname{det}-1)}  \tag{2.65}\\
A_{\left(\begin{array}{cc}
\alpha & \beta \\
\gamma & \delta
\end{array}\right)}|\psi\rangle=U\left(\begin{array}{cc}
\alpha & -\beta \\
\gamma & -\delta
\end{array}\right)|\psi\rangle^{*} \tag{2.66}
\end{gather*}
$$

Here, the matrix $\left(\begin{array}{cc}1 & 0 \\ 0 & -1\end{array}\right)$ is represented by complex conjugation [19]. Given the matrix $F$, we can construct the Unitary matrix

$$
U\left(\begin{array}{ll}
\alpha & -\beta  \tag{2.67}\\
\gamma & -\delta
\end{array}\right)
$$

using the normal construction method of eq. 2.53 and hence we have the action of the anti-unitary operator given a fixed basis.

## Chapter 3

## SIC-POVMs

### 3.1 Quantum Measurements

For the purpose of gaining information from any system, a measurement is necessary. A measurement corresponds to an interaction with the system in a way that we can determine some properties of the system experimentally. For quantum systems, we look towards Quantum Measurements for this task. Though there are ideological differences about the effect of a measurement on the quantum state, everybody agrees that the desired outcome of any measurement is to gain some knowledge of the system. In the POVM formalism, measurements can be described using a set of operators which act on the quantum state in order to give information about it.

### 3.1.1 Quantum States

Physical quantum systems can be described completely using density matrices. A density matrix $\rho$ is a $d$-dimensional matrix acting on the state space of the system which follows

$$
\begin{aligned}
\rho^{\dagger} & =\rho \\
\rho & \geq 0 \\
\operatorname{Tr} \rho & =1
\end{aligned}
$$

The condition $\rho \geq 0$ means that $\rho$ has non-negative eigenvalues. In general, a density matrix can be written as

$$
\rho=\sum_{i} p_{i}\left|\psi_{i}\right\rangle\left\langle\psi_{i}\right|
$$

and is known as a mixed state. If the state of the system can be completely described by a single state vector $|\psi\rangle$, the density operator is given by

$$
\rho=|\psi\rangle\langle\psi|
$$

and is called a pure state. As $\rho|\psi\rangle=|\psi\rangle, 1$ is an eigenvalue of $\rho$. As $\operatorname{Tr} \rho=1$, the rest of the eigenvalues are zero. So the eigenvalues of a pure state are always $(1,0, \ldots, 0)$. Quantum measurements deal with obtaining information about the density matrix of the system. As $\rho$ is Hermitian, we only need to know the upper triangular elements of the matrix to completely determine $\rho$. Also, the diagonal elements are real and are restricted by the condition of $\operatorname{Tr} \rho=1$. The non-diagonal elements are taken to be complex and hence contain two real parameters. Thus, we have in total

$$
(d-1)+\frac{2\left(d^{2}-d\right)}{2}=d^{2}-1
$$

unknown parameters in the density matrix.

### 3.1.2 Partial Trace

For a physical system in a composite dimension $d=d_{1} d_{2}$, the system is described by the density matrix $\rho_{12}$ and the Hilbert space in $\mathbb{C}^{d}$ can be written as a tensor product of the composite dimensions.

$$
\begin{equation*}
\mathcal{H}_{12}=\mathcal{H}_{1} \otimes \mathcal{H}_{2} \tag{3.1}
\end{equation*}
$$

This can be extended to larger systems consisting of many component systems. Given the density matrix in $d$, we can find the state of the system in the component dimensions. Let the basis in $d_{1}$ be given by $\left\{\left|e_{i}\right\rangle\right\}_{i=0}^{d_{1}-1}$ and in $d_{2}$ by $\left\{\left|f_{j}\right\rangle\right\}_{j=0}^{d_{2}-1}$. Then the reduced density matrices $\rho_{1}$ and $\rho_{2}$ in dimensions $d_{1}$ and $d_{2}$ respectively are given by

$$
\begin{aligned}
& \rho_{1}=\operatorname{Tr}_{2} \rho_{12}=\sum_{i=0}^{d_{1}-1}\left\langle e_{i}\right| \rho\left|e_{i}\right\rangle \\
& \rho_{2}=\operatorname{Tr}_{1} \rho_{12}=\sum_{j=0}^{d_{2}-1}\left\langle f_{j}\right| \rho\left|f_{j}\right\rangle
\end{aligned}
$$

where $\rho_{1}$ gives a complete description of the system in the component dimension $d_{1}$ when acted on with operators of the form $A \otimes \mathbb{1}$ and similarly $\rho_{2}$ describes the system in $d_{2}$.

### 3.1.3 POVMs

A general measurement in quantum mechanics can be represented by a positiveoperator valued measure (POVM), which is a set of $m$ operators $E_{i}$ 's obeying

$$
\begin{gather*}
E_{i}=E_{i}^{\dagger}  \tag{3.2}\\
E_{i} \geq 0  \tag{3.3}\\
\sum_{i=1}^{m} E_{i}=\mathbb{1} \tag{3.4}
\end{gather*}
$$

The operators act on the density matrix with the probability that outcome $i$ occurs given by

$$
p_{i}=\operatorname{Tr}\left(E_{i} \rho\right)
$$

Clearly,

$$
\begin{aligned}
\sum_{i} p_{i} & =\sum_{i} \operatorname{Tr}\left(E_{i} \rho\right) \\
& =\operatorname{Tr}\left(\sum_{i} E_{i} \rho\right) \\
& =\operatorname{Tr}(\mathbb{1} \rho) \\
& =1
\end{aligned}
$$

Also, as $\rho$ and $E_{i}$ are positive operators,

$$
\operatorname{Tr}\left(E_{i} \rho\right) \geq 0 \Longrightarrow p_{i} \geq 0
$$

The two conditions

$$
p_{i} \geq 0 \text { and } \sum_{i} p_{i}=1
$$

signify that the $p_{i}$ 's form a probability distribution for the $m$ possible outcomes of the experiment.

A special case of POVM is the Projection-Valued Measure (PVM) where the POVM effects $E_{i}$ 's are given by orthogonal projectors $P_{i}$ 's such that

$$
P_{i} P_{j}=\delta_{i j} P_{i}
$$

and hence the number of operators $m=d$. This is also known as a von Neumann measurement.

As the density matrix has $d^{2}-1$ unknown parameters, an informationally complete POVM will require $d^{2}$ operators. If these operators are symmetric in their trace inner product, we get a POVM which is optimal in a sense relating to quantum state tomography [12][1]. This POVM is hence called a Symmetric Informationally Complete-POVM.

## $3.2 \quad$ SIC-POVMs

A POVM consisting of $d^{2}$ unit vectors $\left|\psi_{I}\right\rangle$ which obey

$$
\begin{equation*}
\frac{1}{d} \sum_{I=1}^{d^{2}}\left|\psi_{I}\right\rangle\left\langle\psi_{I}\right|=\mathbb{1} \tag{3.5}
\end{equation*}
$$

gives us $d^{2}$ operators with one constraint on them, where the elements of the POVM are given by

$$
\begin{equation*}
E_{i}=\frac{1}{d}\left|\psi_{i}\right\rangle\left\langle\psi_{i}\right| \tag{3.6}
\end{equation*}
$$

With these operators acting on our state, we can extract the $d^{2}-1$ independent probabilities $p_{i}$ 's from the density matrix. Recalling that our density matrix has only $d^{2}-1$ unknown parameters, we have obtained the complete information needed to construct the density matrix. Such a POVM is hence called informationally complete. If the vectors also follow

$$
\begin{equation*}
\left|\left\langle\psi_{I} \mid \psi_{J}\right\rangle\right|^{2}=\frac{1}{d+1} \text { where } I \neq J \tag{3.7}
\end{equation*}
$$

the vectors are equiangular and the measure is said to be a symmetric informationally complete POVM or a SIC-POVM.

The $d^{2}$ vectors $\left\{\left|\psi_{I}\right\rangle\right\}_{I=0}^{d^{2}}$ which obey the equations eq. 3.5 and 3.7, are called SIC vectors. Though the equations look simple and harmless, it is extremely difficult to find these $d^{2}$ vectors in higher dimensions. For the exact solutions already found, the components of the vectors are complicated in nature, usually taking numerous pages to write out. There is, however, an interesting property of SIC vectors that makes the task of finding them look less daunting. Most of the SIC solutions already found form an orbit under the Weyl-Heisenberg Group, i.e., given we have an initial vector $\left|\psi_{0}\right\rangle$, the other vectors constituting the POVM are generated by the action of the group elements on the initial vector.

$$
\begin{equation*}
\left|\psi_{i, j}\right\rangle=D_{i, j}\left|\psi_{0}\right\rangle \tag{3.8}
\end{equation*}
$$

For brevity and clarity, a SIC-POVM is simply called a SIC and the vector $\left|\psi_{0}\right\rangle$ generating it a SIC fiducial. As stated earlier, there are $d^{2}$ displacement operators in dimension $d, D_{0,0}$ being the identity, so we will obtain the $d^{2}$ unit vectors using the above construction. The SIC-POVMs which are generated by the action of the Weyl-Heisenberg group on the SIC fiducial vector, are known as WeylHeisenberg SIC-POVMs.

Zauner conjectured that such a fiducial and hence a SIC exists in every finite dimension [13]. This has not been proven yet but is considered true in order to search for SICs in higher dimensions. In fact, every but one solution known till date has been a Weyl-Heisenberg SIC. The oddity is a SIC in dimension 8 found by Hoggar in 1981 [23] which instead of being generated by the Weyl-Heisenberg Group in dimension $8, H(8)$, is generated by $H(2) \times H(2) \times H(2)$. These two groups are not isomorphic as the factors in $H(8)$ are not co-prime and hence Chinese remaindering cannot be applied here.

Even though the solutions in higher dimensions are complicated in nature, the SIC solutions in $d=2,3$ can be figured our analytically without much effort. We will look at these two cases where it is easy to find a SIC given only the relation of completeness and symmetry of inner product.

### 3.2.1 $d=2$

The easiest case is of course, $d=2$. Here, one can find a SIC fiducial by simple calculation using eq. 3.5 , eq. 3.7 and the action of the displacement operators $D_{i, j}^{(2)}$ on the SIC fiducial generating the complete set of vectors. We get the SIC fiducial

$$
\begin{equation*}
\left|\psi_{0}\right\rangle=\frac{1}{\sqrt{6}}\binom{\sqrt{3+\sqrt{3}}}{e^{\frac{i \pi}{4}} \sqrt{3-\sqrt{3}}} \tag{3.9}
\end{equation*}
$$

and the rest of the vectors in the set

$$
\begin{align*}
& \left|\psi_{1}\right\rangle=\frac{1}{\sqrt{6}}\binom{e^{\frac{i \pi}{4}} \sqrt{3-\sqrt{3}}}{\sqrt{3+\sqrt{3}}}  \tag{3.10}\\
& \left|\psi_{2}\right\rangle=\frac{1}{\sqrt{6}}\binom{\sqrt{3+\sqrt{3}}}{-e^{\frac{i \pi}{4}} \sqrt{3-\sqrt{3}}}  \tag{3.11}\\
& \left|\psi_{3}\right\rangle=\frac{i}{\sqrt{6}}\binom{e^{\frac{i \pi}{4}} \sqrt{3-\sqrt{3}}}{-\sqrt{3+\sqrt{3}}} \tag{3.12}
\end{align*}
$$


(a) $\mathrm{SIC}_{1}$

(b) $\mathrm{SIC}_{2}$

Figure 3.1: Representation of the SIC vectors on the Bloch Sphere forming the vertices of dual tetrahedra inscribed within a cube

These vectors are represented by the points on the vertices of a regular tetrahedron inscribed within the Bloch Sphere. In fact, there exists only one other set of SIC vectors in dimension 2, generated by the SIC fiducial,

$$
\begin{equation*}
\left|\psi_{0}^{\prime}\right\rangle=\frac{1}{\sqrt{6}}\binom{-\sqrt{3-\sqrt{3}}}{e^{\frac{i \pi}{4}} \sqrt{3+\sqrt{3}}} \tag{3.13}
\end{equation*}
$$

The set of SIC vectors which are generated by this fiducial form the vertices of a tetrahedron dual relative to the first. Inscribing these tetrahedra within the Bloch Sphere, we get figure 3.1.

### 3.2.2 $d=3$

We can now take a look at the next easiest dimension, $d=3$, where a oneparameter family of fiducials is given by [13] [19]

$$
\left|\psi_{0}\right\rangle=\frac{1}{\sqrt{2}}\left(\begin{array}{c}
0  \tag{3.14}\\
1 \\
e^{i \phi}
\end{array}\right)
$$

Dimension 3 is unique in this sense that we get a family of solutions instead of a few distinct solutions. The SICs in dimensions greater than 3 don't take such simple forms and hence we abstain from listing the exact solutions here. For such cases, we will usually use the numerical solutions in our calculations [24][6]. There are a few exceptions to this rule, particularly in dimension 4 and 7, and we will state these solutions later.

We have now defined SIC-POVMs and looked at some of the solutions in lower dimensions. We can continue the discussion by defining some additional structures which will be useful in our work.

### 3.3 Equiangular Tight Frames

An Equiangular Tight Frame (ETF) is defined as a set of $N$ unit vectors $\left|\Psi_{I}\right\rangle$, where $d \leq N \leq d^{2}$, that follow the condition of being equiangular [25]

$$
\begin{equation*}
\left|\left\langle\Psi_{I} \mid \Psi_{J}\right\rangle\right|^{2}=\frac{N-d}{d(N-1)}, \text { where } I \neq J \tag{3.15}
\end{equation*}
$$

and form a tight frame,

$$
\begin{equation*}
\sum_{I=0}^{N-1}\left|\Psi_{I}\right\rangle\left\langle\Psi_{I}\right|=\frac{N}{d} \mathbb{1} \tag{3.16}
\end{equation*}
$$

Clearly, an ETF is a symmetric POVM which forms a SIC-POVM in the maximal case of having $d^{2}$ vectors. We can also see that for the extreme case of $N=d$, the vectors become orthogonal and the ETF forms an orthonormal basis.

An important property of tight frames can be seen by putting the vectors $\left\{\left|\Psi_{I}\right\rangle\right\}_{I=0}^{N-1}$ as the columns of a matrix $M$. Then,

$$
\begin{align*}
M M^{\dagger} & =\left(\begin{array}{lll}
\left|\Psi_{0}\right\rangle & \ldots & \left|\Psi_{N-1}\right\rangle
\end{array}\right)\left(\begin{array}{c}
\left\langle\Psi_{0}\right| \\
\vdots \\
\left\langle\Psi_{N-1}\right|
\end{array}\right)  \tag{3.17}\\
& =\sum_{I=0}^{N-1}\left|\Psi_{I}\right\rangle\left\langle\Psi_{I}\right| \tag{3.18}
\end{align*}
$$

Now, let the rows of the matrix be denoted by $\left\{\left\langle\Phi_{J}\right|\right\}_{J=0}^{d-1}$.

$$
\begin{align*}
M M^{\dagger} & =\left(\begin{array}{c}
\left\langle\Phi_{0}\right| \\
\vdots \\
\left\langle\Phi_{d-1}\right|
\end{array}\right)\left(\begin{array}{lll}
\left|\Phi_{0}\right\rangle & \ldots & \left.\left|\Phi_{d-1}\right\rangle\right) \\
& =\left(\begin{array}{ccc}
\left\langle\Phi_{0} \mid \Phi_{0}\right\rangle & \ldots & \left\langle\Phi_{0} \mid \Phi_{d-1}\right\rangle \\
\vdots & \ddots & \vdots \\
\left\langle\Phi_{d-1} \mid \Phi_{0}\right\rangle & \ldots & \left\langle\Phi_{d-1} \mid \Phi_{d-1}\right\rangle
\end{array}\right)
\end{array} . . \begin{array}{l} 
\\
\end{array}\right) \tag{3.19}
\end{align*}
$$

Using eq. 3.16 and rescaling the vectors $\left|\Phi_{J}^{\prime}\right\rangle=\sqrt{d / N}\left|\Phi_{J}\right\rangle$

$$
\begin{equation*}
\left\langle\Phi_{I}^{\prime} \mid \Phi_{J}^{\prime}\right\rangle=\delta_{I J} \tag{3.21}
\end{equation*}
$$

So, we see that the rows of the matrix become orthogonal. This property is of utmost importance to this work and one we will use exhaustively in Chapter 6.

ETFs come in use in signal processing [26], communications [26] and coding [27]. In this work, we will use ETFs with suitable choices of $N$ as intermediaries to gain information about SICs in higher dimensions from SICs in lower dimensions.

## $3.4 t$-designs

We will now define another structure, $t$-designs, without getting into much detail. We start with a function $f$ which is homogeneous of degree $t$ in both the components of a Hilbert space vector and the complex conjugate of its components; $f \in \operatorname{Hom}(t, t)$. Then, a finite set of vectors is said to make a t-design if the average of the function over the vectors gives the same value as the average over all of Hilbert space. Although it is not obvious at all, there is a theorem which states that a given set of $K$ unit vectors $\left|\Psi_{I}\right\rangle$ form a $t$-design if they follow the following condition [28] -

$$
\begin{equation*}
\frac{1}{K^{2}} \sum_{I, J=1}^{K}\left|\left\langle\Psi_{I} \mid \Psi_{J}\right\rangle\right|^{2 t}=\frac{t!(d-1)!}{(d-1+t)!} \tag{3.22}
\end{equation*}
$$

Putting $t=1$, we find that the vectors now follow the condition

$$
\begin{equation*}
\frac{1}{K^{2}} \sum_{I, J=1}^{K}\left|\left\langle\Psi_{I} \mid \Psi_{J}\right\rangle\right|^{2}=\frac{1}{d} \tag{3.23}
\end{equation*}
$$

Squaring eq. 3.16 and then taking the trace we will reach the same equation as above, concluding that a POVM is a 1-design. Similarly, for a SIC we square eq. 3.7 and take the sum over the $d^{2}$ vectors giving us

$$
\begin{equation*}
\sum_{I, J=1}^{d^{2}}\left|\left\langle\Psi_{I} \mid \Psi_{J}\right\rangle\right|^{4}=\frac{d^{4}-d^{2}}{(d+1)^{2}}+d^{2}=\frac{2 d^{3}}{d+1} \tag{3.24}
\end{equation*}
$$

Substituting $t=2$ in eq. 3.22 we can check that a SIC is a 2 -design. A consequence of this property of SICs is that we can write

$$
\begin{equation*}
\operatorname{Tr} \rho^{2}=\left\langle\operatorname{Tr} \rho^{2}\right\rangle_{\mathrm{SIC}}=\left\langle\operatorname{Tr} \rho^{2}\right\rangle_{\mathrm{FS}} \tag{3.25}
\end{equation*}
$$

Picking a pure state in the composite dimension $d=m n$, we can take a partial trace to get a reduced density matrix in $\mathbb{C}^{m}, \rho$. The purity is defined by $\operatorname{Tr} \rho^{2}$ and taking an average over all such pure states in $d$, we get the Fubini-Study average given by [28]

$$
\begin{equation*}
\left\langle\operatorname{Tr} \rho^{2}\right\rangle_{\mathrm{FS}}=\frac{m+n}{m n+1} \tag{3.26}
\end{equation*}
$$

This is an important property of density matrices which will come in use later in Chapter 5.

## Chapter 4

## Special case of dimensions $d(d-2)$

Given an exact solution for a SIC, there have been investigations of the number fields that make up these SIC fiducials. We observe that the Weyl-Heisenberg group elements in the representation we have chosen contain only the primitive roots of unity and their powers. Thus, the number field created by the primitive roots of unity, a cyclotomic field, is important in finding the exact solutions of SICs. Another number field which plays a role in the components of the fiducial is the quadratic field [15]. A quadratic field $\mathbb{Q}(\sqrt{D})$, where $D$ is an integer, is the set of all the numbers of the form $x+y \sqrt{D}$. Here $x$ and $y$ are rational numbers whereas $\sqrt{D}$ is irrational. This field is analogous to the complex numbers given by $a+i b$, with the difference that we will insist that $a$ and $b$ are rational. For SICs in dimension $d>3$, the number field that plays a role is given by $\mathbb{Q}(\sqrt{D})$, where

$$
D \text { is the square free part of }(d+1)(d-3)
$$

For example, for $d=7$,

$$
\sqrt{(7+1)(7-3)}=4 \sqrt{2}, \text { so } \sqrt{D}=\sqrt{2}
$$

It should be noted that while the quadratic field $\mathbb{Q}(\sqrt{D})$ is important, it is not enough for the construction of a fiducial. We actually need an extension of the quadratic field in order to write the fiducial components. This extension is in particular difficult for number theorists to understand if $D$ is positive, which is the case here for any $d>3$, compared to when $D$ is negative.

Let us look at a particular fiducial in dimension 7, with Zauner $\left(\begin{array}{cc}2 & 0 \\ 0 & 4\end{array}\right)$

$$
\left|\psi_{0}\right\rangle_{7}=\left(\begin{array}{c}
1  \tag{4.1}\\
\frac{1}{2}(-1-\sqrt{2}+\sqrt{-1+2 \sqrt{2}}) \\
\frac{1}{2}(-1-\sqrt{2}+\sqrt{-1+2 \sqrt{2}}) \\
\frac{1}{2}(-1-\sqrt{2}-\sqrt{-1+2 \sqrt{2}}) \\
\frac{1}{2}(-1-\sqrt{2}+\sqrt{-1+2 \sqrt{2}}) \\
\frac{1}{2}(-1-\sqrt{2}-\sqrt{-1+2 \sqrt{2}}) \\
\frac{1}{2}(-1-\sqrt{2}-\sqrt{-1+2 \sqrt{2}})
\end{array}\right)
$$

Here, we have ignored an overall normalization factor. We chose this fiducial as its exact solution is relatively simple to write, being one of the rare few fiducials in dimensions greater than 3 which can be written in such an easy way. From this fiducial we observe that while the quadratic field created by $\sqrt{2}$ is important, another number which plays a role here is $\sqrt{-1+2 \sqrt{2}}$, which obviously belongs to an extension of $\mathbb{Q}(\sqrt{2})$. We can also observe from the simple structure of the fiducial that our previous argument in Section 2.7 considering the Zauner unitary being simpler in the case of $\beta=0$ is correct. The solution given by Scott and Grassl (2010) [24] uses the standard form of the Zauner, and is therefore more complicated.

Now that we have established to a degree that number theory makes an important contribution, we would like to use it to our advantage in our search for SICs. Regarding the quadratic field $\mathbb{Q}(\sqrt{D})$, we find that there is a special connection between the quadratic field in dimension $d$ and in dimension $d(d-2)$. Looking at the field for $d(d-2)$

$$
\begin{aligned}
D & =\text { the square free part of }(d(d-2)+1)(d(d-2)-3) \\
& =\text { the square free part of }(d-1)^{2}(d+1)(d-3) \\
& =\text { the square free part of }(d+1)(d-3)
\end{aligned}
$$

So, the quadratic field which is used while writing the fiducial is the same in dimension $d$ and $d(d-2)$, giving rise to similarities in the SICs in these particular dimensions. There is more to the connection between the number fields in these two dimensions, but for that we refer to Appleby et al. (2017) [3]

Even though the SICs might be similar in construction, we would like to know if there is an explicit connection between the SICs in the two dimensions. This
connection comes in the form of an argument concerning overlap phases.

We define the overlap phase factors in dimension $d$ as $\theta_{i, j}$ where

$$
e^{i \theta_{i, j}}= \begin{cases}1 & i=j=0  \tag{4.2}\\ \sqrt{d+1}\left\langle\psi_{0}\right| D_{i, j}\left|\psi_{0}\right\rangle & \text { otherwise }\end{cases}
$$

Given a SIC fiducial, we can calculate these $d^{2}$ phase factors. As a result of the symmetries of the fiducials, we generally have less than $d^{2}$ independent phase factors. We will investigate the effect of symmetries on SIC fiducials and pursue the restrictions in more detail in Chapter 5. Now, looking at the overlap phases for a SIC fiducial $\left|\Psi_{0}\right\rangle$ in dimension $d(d-2)$ as well,

$$
e^{i \Theta_{i, j}}= \begin{cases}1 & i=j=0  \tag{4.3}\\ \sqrt{d(d-2)+1}\left\langle\Psi_{0}\right| D_{i, j}\left|\Psi_{0}\right\rangle & \text { otherwise }\end{cases}
$$

Appleby et al. (2017) [18] give the following relation

$$
\begin{equation*}
e^{i \Theta_{i, j}}= \pm e^{2 i \theta_{i^{\prime}, j^{\prime}}} \tag{4.4}
\end{equation*}
$$

where $\left(i^{\prime}, j^{\prime}\right)$ is linearly related to $(i, j)$, which is verified empirically for the cases looked at in the paper. One of the main aims of this thesis is to be able to get this relation as a natural outcome of the connection. Another observation given in the paper for when $d$ is odd is

$$
\begin{equation*}
\sqrt{d(d-2)+1}\left\langle\Psi_{0}\right| D_{d i, d j}\left|\Psi_{0}\right\rangle=+1 \tag{4.5}
\end{equation*}
$$

This connection for odd $d$ is verified in Chapter 6.

This interconnection goes further from $d(d-2)$ to $d(d-2)(d(d-2)-2)$ and so on. For example, the following SIC fiducials are connected in such a manner,

$$
\begin{equation*}
5 a \rightarrow 15 d \rightarrow 195 d \tag{4.6}
\end{equation*}
$$

These connected fiducials are called aligned. Here, and henceforth in the text, the SIC fiducials are taken from and labelled according to Scott and Grassl (2010) [24] and Scott (2017) [6]. This progression in dimensions is called "laddering" by Appleby et al. (2017) and has been observed in many dimensions greater than 3. There also exist fiducials 195a, 195b and 195c which are aligned to the fiducials $15 \mathrm{a}, 15 \mathrm{~b}$ and 15 c respectively.

While on the topic of laddering, we observe that given a dimension $d$, it takes a maximum of two 'steps' to reach a dimension divisible by 3 .

$$
\begin{align*}
& d=0 \bmod 3 \rightarrow 0 \bmod 3 \rightarrow 0 \bmod 3  \tag{4.7}\\
& d=1 \bmod 3 \rightarrow 2 \bmod 3 \rightarrow 0 \bmod 3  \tag{4.8}\\
& d=2 \bmod 3 \rightarrow 0 \bmod 3 \rightarrow 0 \bmod 3 \tag{4.9}
\end{align*}
$$

This suggests that dimensions divisible by 3 deserve special attention and we will look at some particular dimensions of the form $d=3 k$ in the next chapter.

## Chapter 5

## Dimensions of the form $d=3 k$

### 5.1 Symmetries

Having discussed the preliminaries of SICs, it is useful to talk about the symmetries present in different dimensions. The knowledge of symmetries is fundamental to understanding the different SIC solutions and the connection symmetries have with the dimension. We will make use of the Clifford Group discussed earlier in Section 2.7. The importance of the Clifford Group in the study of SICs will become abundantly clear in the course of this section.

### 5.1.1 Zauner Symmetry

We can start the discussion by stating Zauner's conjecture [19] which states that in every finite dimension there exists a SIC fiducial, which is an eigenvector of an order 3 Clifford unitary. One such example of this is constructed from the symplectic matrix

$$
Z=\left(\begin{array}{ll}
0 & -1  \tag{5.1}\\
1 & -1
\end{array}\right)
$$

Using equation 2.53 we can now construct the unitary matrix in any dimension $d$ such that

$$
\begin{equation*}
U_{Z}\left|\psi_{0}\right\rangle=\left|\psi_{0}\right\rangle \tag{5.2}
\end{equation*}
$$

We call this unitary a Zauner. Appleby (2005) further conjectured that every Weyl-Heisenberg SIC fiducial is an eigenvector of a canonical order 3 unitary [19]. He defines a canonical order 3 unitary by an operation $D_{\mathbf{p}} U_{F}$ if

1. $\operatorname{Tr}(\mathrm{F})=-1(\bmod d)$
2. F is not the identity matrix

The second condition exists only to exclude the identity matrix in the unique case of $d=3$ as $\operatorname{Tr}(\mathbb{1})=-1(\bmod 3)$. As is the case, for prime dimensions excluding 3 , any symplectic matrix F with trace -1 is conjugate to the Zauner, i.e. there exists a matrix $C \in \mathrm{SL}\left(2, \mathbb{Z}_{d}\right)$ such that

$$
\begin{equation*}
C Z C^{-1}=F \tag{5.3}
\end{equation*}
$$

Clearly, the trace remains the same and we get a canonical order 3 unitary. In fact, for prime $d$, the trace uniquely labels the conjugacy class unless $\operatorname{Tr}= \pm 2$ $(\bmod d)$.

This in turn inspires us to find a basis in which the fiducial looks simpler. We only allow coordinate changes that retain the representation of the Weyl-Heisenberg Group. Recalling that the unitary is simpler when $\beta=0$, we can try to find cases when in prime $d$, there exists such a matrix conjugate to $Z$.

$$
\begin{gather*}
F=\left(\begin{array}{cc}
\alpha & 0 \\
\gamma & -1-\alpha
\end{array}\right)  \tag{5.4}\\
-\alpha-\alpha^{2}=1(\bmod d)  \tag{5.5}\\
\alpha^{2}+\alpha+1=0(\bmod d)  \tag{5.6}\\
\left(\alpha+2^{-1}\right)^{2}=2^{-2}-1(\bmod d) \tag{5.7}
\end{gather*}
$$

So, this is possible only when $2^{-2}-1$ is a quadratic residue $(\bmod d)$. In particular, this happens when $d=1 \bmod 3$. One such example can be given in dimension 7, where

$$
\left(\begin{array}{ll}
2 & 0  \tag{5.8}\\
0 & 4
\end{array}\right) \text { is conjugate to }\left(\begin{array}{ll}
0 & -1 \\
1 & -1
\end{array}\right)
$$

Making the change of basis such that the fiducial is an eigenvector of the unitary constructed from the former matrix, we get a much simpler form of the fiducial, as can be seen in eq. 4.1.

The Zauner symmetry given by eq. 5.1 occurs in every dimension for the known SICs with some exceptions occuring in dimensions of the form $d=3(3 k+1)$, where the symmetry is given by [24]

$$
F_{a}=\left(\begin{array}{cc}
1 & d+3  \tag{5.9}\\
d+3 k & d-2
\end{array}\right)
$$

These matrices are order-3 unitaries known as Zauner matrices type $F_{a}$. Chinese Remaindering on $F_{a}$ gives us an identity matrix in the dimension 3 factor tensor product with an order-3 matrix in dimension $3 k+1$.

### 5.1.2 Additional Symmetries

In addition to the Zauner symmetry, there are other symmetries present for some dimensions. As we focus on dimensions of the form

$$
\begin{equation*}
d(d-2)=(d-1)^{2}-1 \tag{5.10}
\end{equation*}
$$

in this thesis, we look at a particular order- 2 symmetry present in dimensions of the type $N=k^{2}-1$, given by the symplectic matrix [24]

$$
F_{b}=\left(\begin{array}{cc}
-k & N  \tag{5.11}\\
N & N-k
\end{array}\right)
$$

For odd dimensions, $d$ and $d-2$ are relatively prime and hence we can apply the Chinese Remainder Theorem on $F_{b}$ using eq. 2.62.

$$
\left(\begin{array}{cc}
1-d & 0  \tag{5.12}\\
0 & 1-d
\end{array}\right)_{d(d-2)}=\left(\begin{array}{cc}
-1 & 0 \\
0 & -1
\end{array}\right)_{d-2} \otimes\left(\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right)_{d}
$$

The symplectic matrix for $(d-2)$ gives the unitary matrix $U_{P}$ which is called the parity operator. For odd dimensions, $U_{P}$ has spectrum $((d+1) / 2,(d-1) / 2)$. Thus, a SIC in dimension $d(d-2)$ has a symmetry of the form

$$
\begin{equation*}
U_{P}^{(d-2)} \otimes \mathbb{1}^{(d)} \tag{5.13}
\end{equation*}
$$

### 5.1.3 Centered Fiducials

A SIC fiducial is said to be centered if it is an eigenvector of a pure symplectic, i.e., $U_{F}|\psi\rangle=|\psi\rangle$ instead of an ordinary canonical order 3 unitary

$$
\begin{equation*}
D_{\mathbf{p}} U_{F}\left|\psi_{0}\right\rangle=\left|\psi_{0}\right\rangle \tag{5.14}
\end{equation*}
$$

Though centered SIC fiducials can be found in numerous dimensions, it is still an unanswered question if every individual SIC has a centered fiducial. Given a fiducial which follows eq. 5.14, we want a shifted fiducial $\left|\psi_{q}\right\rangle=D_{\mathbf{q}}\left|\psi_{0}\right\rangle$ such that

$$
\begin{equation*}
U_{F}\left|\psi_{q}\right\rangle=\left|\psi_{q}\right\rangle \tag{5.15}
\end{equation*}
$$

$$
\begin{gather*}
U_{F} D_{\mathbf{q}}\left|\psi_{0}\right\rangle=U_{F} D_{\mathbf{q}} U_{F}^{\dagger} U_{F}\left|\psi_{0}\right\rangle=D_{F \mathbf{q}} U_{F}\left|\psi_{0}\right\rangle  \tag{5.16}\\
D_{F \mathbf{q}} U_{F}\left|\psi_{0}\right\rangle=D_{\mathbf{q}}\left|\psi_{0}\right\rangle  \tag{5.17}\\
U_{F}\left|\psi_{0}\right\rangle=D_{(1-F) \mathbf{q}}\left|\psi_{0}\right\rangle  \tag{5.18}\\
D_{\mathbf{p}} U_{F}\left|\psi_{0}\right\rangle=\left|\psi_{0}\right\rangle=D_{\mathbf{p}} D_{(1-F) \mathbf{q}}\left|\psi_{0}\right\rangle  \tag{5.19}\\
\Rightarrow D_{-\mathbf{p}}\left|\psi_{0}\right\rangle=D_{(1-F) \mathbf{q}}\left|\psi_{0}\right\rangle \tag{5.20}
\end{gather*}
$$

This condition simply gives us that given $F$ and $\mathbf{p}, \mathbf{q}$ should satisfy

$$
\begin{equation*}
\mathbf{p}=(F-1) \mathbf{q} \tag{5.21}
\end{equation*}
$$

When examining this equation, one finds that it can be solved for $\mathbf{q}$ given $\mathbf{p}$ and $F$ if the dimension is not divisible by 3 . This shows that in dimensions not divisible by 3 every SIC contains a centered fiducial. The case of dimensions divisible by 3 , however, remain open by this argument.

### 5.2 Dimensions $d=3 k$

We have established that the laddering of dimensions eventually leads us to dimensions which are divisible by 3 . We look at some specific examples in the ladder to find out their properties. For dimensions of the form $d=3 k$, we can take a partial trace to get a reduced density matrix.

$$
\rho_{3}=\operatorname{Tr}_{k} \rho
$$

For example, taking the numerical SIC fiducial $\left|\psi_{15}\right\rangle=15 d$ [24], we take the partial trace on the density matrix formed.

$$
\begin{gather*}
\rho=\left|\psi_{15}\right\rangle\left\langle\psi_{15}\right|  \tag{5.22}\\
\rho_{3}=\operatorname{Tr}_{5} \rho \tag{5.23}
\end{gather*}
$$

We get the following reduced density matrix in dimension 3

$$
\rho_{3}=\left(\begin{array}{ccc}
\frac{1}{2} & 0 & 0  \tag{5.24}\\
0 & \frac{1}{4} & \frac{1}{4} \\
0 & \frac{1}{4} & \frac{1}{4}
\end{array}\right)
$$

Similarly, we can get a reduced density matrix in dimension 3 given any SIC vector in a higher dimension $d=3 k$. In fact, for dimensions of the form $d=3 k$ with the standard Zauner symmetry $F_{Z}$ (eq. 5.1), the reduced density matrix in dimension 3 can always be brought to a simpler form by a change of basis

$$
\rho_{3}=\left(\begin{array}{ccc}
a_{11} & 0 & 0  \tag{5.25}\\
0 & a_{22} & a_{23} \\
0 & a_{32} & a_{33}
\end{array}\right)
$$

For some exceptional cases when $d=3(3 k+1)$ and the Zauner symmetry is $F_{a}$ (eq. 5.9), it is unclear how to bring the reduced density matrix to this form. Let the eigenvalues of this reduced matrix be $\lambda_{1}, \lambda_{2}, \lambda_{3}$. For some special cases of the laddering dimensions, $15 d, 48 \mathrm{~g}$ and $195 d$ [24][6] the eigenvalues follow

$$
\begin{aligned}
\lambda_{2} & =\lambda_{3} \\
\Rightarrow \lambda_{1}+2 \lambda_{2} & =1 \text { as } \operatorname{Tr} \rho_{3}=1
\end{aligned}
$$

Recalling the property of SICs in eq 3.26 and putting $d=3 k$ these eigenvalues also follow,

$$
\begin{equation*}
\lambda_{1}^{2}+2 \lambda_{2}^{2}=\frac{k+3}{3 k+1} \tag{5.26}
\end{equation*}
$$

This gives us the value for $\lambda_{1}$ and $\lambda_{2}$ as

$$
\begin{gather*}
\lambda_{2}=\frac{\sqrt{d+1} \pm 2}{3 \sqrt{d+1}}  \tag{5.27}\\
\lambda_{1}=1-2 \lambda_{2} \tag{5.28}
\end{gather*}
$$

We can see that as eq. 5.26 is quadratic, we get two sets of eigenvalues. This calculation doesn't provide further information as to which solution should be chosen. There is, however, certainly a choice to be made as can be seen from the reduced density matrices in dimension 3 given the SIC in dimensions $15 d, 48 \mathrm{~g}$ and $195 d$ which are of the form

$$
\rho_{3}=\left(\begin{array}{ccc}
1-2 a & 0 & 0  \tag{5.29}\\
0 & a & 1-3 a \\
0 & 1-3 a & a
\end{array}\right)
$$

|  |  |
| :---: | :---: |
| SIC | a |
| 15 d | $\frac{1}{4}$ |
| 48 g | $\frac{2}{7}$ |
| 195 d | $\frac{5}{14}$ |

Table 5.1: Values of a for Reduced Density Matrices

| SIC | Eigenvalues |
| :---: | :---: |
| 15 d | $0, \frac{1}{2}, \frac{1}{2}$ |
| 48 g | $\frac{1}{7}, \frac{3}{7}, \frac{3}{7}$ |
| 195 d | $\frac{3}{7}, \frac{2}{7}, \frac{2}{7}$ |

Table 5.2: Eigenvalues of Reduced Density Matrices

For the given SICs, $15 d$ and $48 g$ take the greater value of $\lambda_{2}$, i.e., the positive sign is chosen in the equation 5.27. 195d however, takes smaller value of $\lambda_{2}$ given by the negative sign. We can see that the reduced matrices formed using the SICs above are not pure states as, for a pure state the eigenvalues should be $(1,0,0, \ldots, 0)$. We can however create a set of special density matrices using the reduced matrix we have,

$$
\begin{equation*}
\rho(x)=x \rho_{3}+\frac{(1-x)}{3} \mathbb{1}_{3} \tag{5.30}
\end{equation*}
$$

where $x$ is a variable bounded by the condition that $\rho(x)$ has positive eigenvalues. The matrices created by such a construction can be thought to lie on a line with $x$ as their dependent variable. For $x=1$, we get our reduced density matrix back and for $x=0$, we have the identity matrix in dimension 3 with trace 1 . We can check that the matrix in 5.30 has trace 1 and so forms a density matrix. To find the points where the line hits the boundary of the set of density matrices, we can vary $x$ till the determinant reaches 0 . As $x$ can take positive as well as negative values, we have an upper as well as a lower bound. If for one of the boundary matrices, we get a pure state, we know that it is a SIC. Looking specifically at
the cases for SICs $15 d, 48 g$ and $195 d$, we find that for the density matrix $\rho(x)$

$$
\begin{equation*}
\operatorname{det} \rho(x)=\frac{1}{27}\left(1-\frac{12 x^{2}}{(\sqrt{d+1})^{2}} \pm \frac{16 x^{3}}{(\sqrt{d+1})^{3}}\right) \tag{5.31}
\end{equation*}
$$

Here again, $195 d$ behaves differently from the other two as it takes the positive sign at the end of the expression whereas $15 d$ and $48 g$ take the negative.

If we now look at the solutions to

$$
\operatorname{det} \rho(x)=0
$$

we can find the bound on $x$. For the 3 cases we have taken to look into detail, this happens for

| SIC | $x$ |
| :---: | :---: |
| 15 d | $-2,-2,1$ |
| 48 g | $-\frac{7}{2},-\frac{7}{2},-\frac{7}{4}$ |
| 195 d | $-\frac{7}{2}, 7,7$ |

Table 5.3: Values of x for determinant 0

Incidentally, we get three solutions for $x$ as eq 5.31 is cubic. To get the density matrix, we put the value of $x$ in eq 5.30 . We find that for the above examples, we get a pure state for the repeated roots of $x .195 d$ is the oddity again as the value of $x$ for which we get a pure state is positive, i.e., 7 . In cases $15 d$ and $48 g$, $x$ is -2 and -3.5 respectively for $\rho(x)$ to have eigenvalues $(1,0,0)$.

Looking at the reduced density matrix in dimension 3 for SICs in $15 d, 48 g$ and $195 d$ again (eq. 5.29), the line

$$
\begin{equation*}
\rho(x)=x \rho_{3}+\frac{(1-x)}{3} \mathbb{1}_{3} \tag{5.32}
\end{equation*}
$$

ends on the boundary as a special pure state $|\psi\rangle\langle\psi|$ where

$$
|\psi\rangle=\frac{1}{\sqrt{2}}\left(\begin{array}{c}
0  \tag{5.33}\\
1 \\
-1
\end{array}\right)
$$

which is a SIC in dimension 3 for the value of x

$$
\begin{equation*}
x=\frac{-1}{2(1-3 a)} \tag{5.34}
\end{equation*}
$$

We see that this will work as long as $a \neq 1 / 3$ and the values of $x$ for SICs $15 d, 48 g$ and $195 d$ correspond to the ones we got before.

| SIC | x |
| :---: | :---: |
| 15 d | -2 |
| 48 g | $-\frac{7}{2}$ |
| 195 d | 7 |

Table 5.4: Values of x for a SIC in dimension 3

## Chapter 6

## Constructing ETFs from SICs

We have now looked at the connection between SICs in dimensions $d$ and $d(d-$ 2) and observed some satisfactory relations. The ultimate goal, however, is to construct a SIC in dimension $d(d-2)$ given only the SIC in dimension $d$. As noted before, a construction performing this would allow us to recursively create SICs in higher dimensions, thus coming one step closer to finding a proof of existence for SICs. We give the procedure in an elaborate series of steps which mainly consist of first creating an ETF from a known SIC using a construction given by Renes et al. (2004) [12], and secondly creating new ETFs from this constructed ETF in particular dimensions using a method known as the Naimark Extension Theorem.

### 6.1 Naimark Theorem

The Naimark Theorem states that given an ETF in dimension $d$ with $N$ vectors, there exists an ETF in dimension $N-d$ with $N$ vectors, given $N>d$ [29]. From here on, we shall denote an ETF by its dimension followed by its size. So, the construction gives

$$
\begin{equation*}
\operatorname{ETF}_{(d, N)} \Rightarrow \operatorname{ETF}_{(N-d, N)} \tag{6.1}
\end{equation*}
$$

Let us look at the exact way to do this. Given an $\operatorname{ETF}_{(d, N)}$, if we put the vectors of the ETF as a column in a matrix, the rows are orthogonal to each other (eq 3.21) and can be normalized up to a common factor. As these $d$ vectors are orthogonal, there must exist $N-d$ vectors in dimension $N$ which complete the basis. Completing the basis forms a $(N-d) \times N$ matrix which is called the Naimark Complement. Putting the complete basis vectors as the rows of a
matrix, we get a unitary matrix of dimension $N$.

$$
U=\left(\begin{array}{c}
\left\langle x_{1}\right|  \tag{6.2}\\
\vdots \\
\frac{\left\langle x_{d}\right|}{\left\langle x_{d+1}\right|} \\
\vdots \\
\left\langle x_{N}\right|
\end{array}\right)_{N \times N}=\left(\begin{array}{ccc}
\left|u_{1}\right\rangle & \ldots & \left|u_{N}\right\rangle \\
\hline\left|v_{1}\right\rangle & \ldots & \left|v_{N}\right\rangle
\end{array}\right)_{N \times N}
$$

Here, the ETF vectors are given by $\left\{\left|u_{I}\right\rangle\right\}_{I=1}^{N}$. As the columns of a unitary matrix are also orthogonal,

$$
\begin{gather*}
\left\langle u_{I} \mid u_{J}\right\rangle+\left\langle v_{I} \mid v_{J}\right\rangle=0 \text { for } I \neq J  \tag{6.3}\\
\left\langle v_{I} \mid v_{J}\right\rangle=-\left\langle u_{I} \mid u_{J}\right\rangle \tag{6.4}
\end{gather*}
$$

With this, we get that the vectors $\left\{\left|v_{I}\right\rangle\right\}_{I=1}^{N}$ are equiangular. As the rows of the lower matrix are orthogonal, it also satisfies the condition of being a tight frame. Hence, the vectors $\left|v_{I}\right\rangle$ form an ETF in dimension $N-d$ of size $N$.

$$
U=\left(\begin{array}{ccc} 
& &  \tag{6.5}\\
\cdots & \operatorname{ETF}_{(d, N)} & \cdots \\
& & \\
\cdots & \operatorname{ETF}_{(N-d, N)} & \cdots
\end{array}\right)_{N \times N}
$$

We should keep in mind that given a set of orthonormal vectors, there is no unique way to complete the basis. Hence, there are various different ways to create the Naimark Complement. However, we will look at certain desirable properties of the Complement later to get a favourable solution.

### 6.2 Naimark Complement of a SIC

Let us look at the construction of the Naimark Complement of a SIC as an exercise into understanding more about the properties of the Complement. We know how to get the complete set of SIC vectors from the SIC fiducial vector. If we have the SIC fiducial $\left|\psi_{0}\right\rangle$ in dimension $d$, we act on it with $D_{i, j}$ where $0 \leq i, j \leq d-1$.

Putting these vectors in a matrix as columns we get

$$
M_{1}=\left(\begin{array}{cccc}
\mid & & &  \tag{6.6}\\
& & & \\
\psi_{0} & D_{0,1} \psi_{0} & \ldots & D_{d-1, d-1} \psi_{0} \\
\mid & \mid & &
\end{array}\right)_{d \times d^{2}}
$$

A SIC is an ETF in dimension $d$ of size $d^{2}$, so we can apply the Naimark Theorem to create another ETF in dimension $d^{2}-d$ with $d^{2}$ vectors from the matrix $M_{1}$. As there is no unique way of creating such a complement matrix, we can now require the vectors to be generated by the Weyl-Heisenberg group.

Let us denote the fiducial of the $\operatorname{ETF}_{\left(d, d^{2}\right)}$ as $\left|x_{1}\right\rangle_{d}$.

$$
\mathrm{ETF}_{1}=\left(\begin{array}{llll}
\left|x_{1}\right\rangle & D_{0,1}\left|x_{1}\right\rangle & \ldots & D_{d-1, d-1}\left|x_{1}\right\rangle \tag{6.7}
\end{array}\right)_{d \times d^{2}}
$$

We now choose $d-1$ vectors, $\left|x_{2}\right\rangle_{d} \ldots\left|x_{d}\right\rangle_{d}$, and construct the column matrix,

$$
\left(\begin{array}{c}
\left|x_{2}\right\rangle_{d}  \tag{6.8}\\
\left|x_{3}\right\rangle_{d} \\
\vdots \\
\left|x_{d}\right\rangle_{d}
\end{array}\right)
$$

We act on this column vector with $\mathbb{1}^{(d-1)} \otimes D_{\mathbf{p}}^{(d)}$ to get

$$
\mathrm{ETF}_{2}=\left(\begin{array}{cccc}
\left|x_{2}\right\rangle & D_{0,1}\left|x_{2}\right\rangle & \ldots & D_{d-1, d-1}\left|x_{2}\right\rangle  \tag{6.9}\\
\vdots & \vdots & & \vdots \\
\left|x_{d}\right\rangle & D_{0,1}\left|x_{d}\right\rangle & \ldots & D_{d-1, d-1}\left|x_{d}\right\rangle
\end{array}\right)_{d^{2}-d \times d^{2}}
$$

The square matrix thus formed is

$$
\mathrm{S}=\left(\begin{array}{cccc}
\left|x_{1}\right\rangle & D_{0,1}\left|x_{1}\right\rangle & \ldots & D_{d-1, d-1}\left|x_{1}\right\rangle  \tag{6.10}\\
\hline\left|x_{2}\right\rangle & D_{0,1}\left|x_{2}\right\rangle & \ldots & D_{d-1, d-1}\left|x_{2}\right\rangle \\
\vdots & \vdots & & \vdots \\
\left|x_{d}\right\rangle & D_{0,1}\left|x_{d}\right\rangle & \ldots & D_{d-1, d-1}\left|x_{d}\right\rangle
\end{array}\right)_{d^{2} \times d^{2}}
$$

Computing $S S^{\dagger}$

$$
\begin{align*}
S S^{\dagger} & =\sum_{\mathbf{p}} D_{\mathbf{p}} \mathbf{x}_{i} \mathbf{x}_{j}^{\dagger} D_{\mathbf{p}}^{\dagger}  \tag{6.11}\\
& =d\left\langle x_{i} \mid x_{j}\right\rangle \mathbb{1}_{d} \tag{6.12}
\end{align*}
$$

as the Weyl-Heisenberg Group forms a unitary operator basis [21]. For $S$ to be a unitary matrix, we require

$$
\begin{equation*}
\left\langle x_{i} \mid x_{j}\right\rangle=\frac{1}{d} \delta_{i j} \tag{6.13}
\end{equation*}
$$

So, by constructing vectors $\left|x_{2}\right\rangle_{d} \ldots\left|x_{d}\right\rangle_{d}$ in such a way that they obey 6.13, we can create the Naimark complement of the SIC generator matrix. In order for the two matrix blocks to be considered as ETFs, the vectors will have to be rescaled accordingly. The $\operatorname{ETF}_{\left(d^{2}-d, d^{2}\right)}$ constructed from a $\operatorname{SIC}_{\left(d, d^{2}\right)}$ is called a dual SIC.

Let us now look at some examples of this construction. We are skipping the case for $d=2$ as it is self-dual, i.e., the dual SIC we get is the SIC itself.

### 6.2.1 Dual SIC in dimension 3

We take the case of $d=3$ as an easy example as the fiducial has a simple form. The generator matrix for the SIC is given by taking the SIC fiducial and acting on it by $D_{\mathrm{p}}^{(3)}$

$$
\begin{gathered}
\mathbf{x}_{1}=\frac{1}{\sqrt{2}}\left(\begin{array}{c}
0 \\
1 \\
e^{i \phi}
\end{array}\right) \\
\operatorname{ETF}_{(3,9)}=\frac{1}{\sqrt{2}}\left(\begin{array}{ccccccccc}
0 & e^{i \phi} & 1 & 0 & \omega e^{i \phi} & \omega^{2} & 0 & \omega^{2} e^{i \phi} & \omega \\
1 & 0 & e^{i \phi} & \omega & 0 & e^{i \phi} & \omega^{2} & 0 & e^{i \phi} \\
e^{i \phi} & 1 & 0 & \omega^{2} e^{i \phi} & 1 & 0 & \omega e^{i \phi} & 1 & 0
\end{array}\right)
\end{gathered}
$$

where

$$
\omega=e^{\frac{2 i \pi}{3}} \text { for } d=3
$$

We find vectors $\mathbf{x}_{\mathbf{2}}$ and $\mathbf{x}_{\mathbf{3}}$ such that $\mathbf{x}_{\mathbf{1}}, \mathbf{x}_{\mathbf{2}}$ and $\mathbf{x}_{\mathbf{3}}$ form an orthonormal basis

$$
\mathbf{x}_{2}=\frac{1}{2}\left(\begin{array}{c}
\sqrt{2} \\
-1 \\
e^{i \phi}
\end{array}\right) ; \quad \mathbf{x}_{3}=\frac{1}{2}\left(\begin{array}{c}
-\sqrt{2} \\
-1 \\
e^{i \phi}
\end{array}\right)
$$

Now, putting these together to form a vector and rescaling accordingly, we get

$$
\frac{1}{2 \sqrt{2}}\left(\begin{array}{c}
\sqrt{2} \\
-1 \\
e^{i \phi} \\
-\sqrt{2} \\
-1 \\
e^{i \phi}
\end{array}\right)
$$

To get the Naimark Complement, we act on this vector with

$$
\mathbb{1}^{(2)} \otimes D_{\mathbf{p}}^{(3)}=\left(\begin{array}{cc}
D_{\mathbf{p}}^{(3)} & \mathbf{0} \\
\mathbf{0} & D_{\mathbf{p}}^{(3)}
\end{array}\right)
$$

We then get the Dual-SIC, $\operatorname{ETF}_{(6,9)}$

$$
\begin{aligned}
& \operatorname{ETF}_{(6,9)}= \\
& \frac{1}{2 \sqrt{3}}\left(\begin{array}{ccccccccc}
\sqrt{2} & e^{i \phi} & -1 & \sqrt{2} & \omega e^{i \phi} & -\omega^{2} & \sqrt{2} & \omega^{2} e^{i \phi} & -\omega \\
-1 & \sqrt{2} & e^{i \phi} & -\omega & \sqrt{2} \omega^{2} & e^{i \phi} & -\omega^{2} & \sqrt{2} \omega & e^{i \phi} \\
e^{i \phi} & -1 & \sqrt{2} & \omega^{2} e^{i \phi} & -1 & \sqrt{2} \omega & \omega e^{i \phi} & -1 & \sqrt{2} \omega^{2} \\
-\sqrt{2} & e^{i \phi} & -1 & -\sqrt{2} & \omega e^{i \phi} & -\omega^{2} & -\sqrt{2} & \omega^{2} e^{i \phi} & -\omega \\
-1 & -\sqrt{2} & e^{i \phi} & -\omega & -\sqrt{2} \omega^{2} & e^{i \phi} & -\omega^{2} & -\sqrt{2} \omega & e^{i \phi} \\
e^{i \phi} & -1 & -\sqrt{2} & \omega^{2} e^{i \phi} & -1 & -\sqrt{2} \omega & \omega e^{i \phi} & -1 & -\sqrt{2} \omega^{2}
\end{array}\right)
\end{aligned}
$$

We can check that the column vectors of the above matrix satisfy the equations 3.15 and 3.16 and hence form an ETF.

Constructing Dual SICs in higher dimensions becomes a bit cumbersome but it follows the same form and hence we can always get an $\operatorname{ETF}_{\left(d^{2}-d, d^{2}\right)}$ from a $\operatorname{SIC}_{\left(d, d^{2}\right)}$.

### 6.3 Constructing $\operatorname{ETF}_{\left(\frac{d(d+1)}{2}, d^{2}\right)}$ from SIC

Coming back to the problem of constructing SICs in dimension $d(d-2)$ using a SIC in dimension $d$, we look at the steps to be taken towards this goal. We will construct new intermediate ETFs using a given SIC fiducial in dimension $d$ and finally connect them to the SIC in dimension $d(d-2)$.

The first step is to construct an ETF in dimension $d(d+1) / 2$ having $d^{2}$ vectors, from a SIC in dimension $d$, the construction for which was first given by Renes et al. (2004) [12] and more recently by Ostrovskyi and Yakymenko (2019)
[30].
Theorem 1. The vectors $\left\{\left|\psi_{i}\right\rangle\right\}_{i=0}^{d^{2}} \in \mathbb{C}^{d}$ form a SIC iff i) $\left\{\left|\psi_{i}\right\rangle\right\}_{i=0}^{d^{2}}$ form a tight frame in $\mathbb{C}^{d}$
ii) $\left\{\left|\psi_{i}\right\rangle \otimes\left|\psi_{i}\right\rangle\right\}_{i=0}^{d^{2}}$ form a tight frame in $\mathbb{C}^{d} \otimes_{S} \mathbb{C}^{d}$

Proof. The proof of this theorem follows from Proposition 2 and 3 in Ostrovskyi and Yakymenko (2019) [30].

Instead of the cumbersome subscript in $\operatorname{ETF}_{\left(\frac{d(d+1)}{2}, d^{2}\right)}$, we will simply call this ETF $_{1}$ from now. The dimension $d$ will be apparent according to the discussion and will be stated clearly in case it is needed to avoid confusion.

Given a SIC fiducial $\left|\psi_{0}\right\rangle$ in dimension $d$, the product vector $\left|\psi_{0}\right\rangle \otimes\left|\psi_{0}\right\rangle$ sits in a symmetric subspace of dimension $d(d+1) / 2$. For example, for a fiducial vector in $d=3$

$$
\begin{align*}
\left|\psi_{0}\right\rangle=\left(\begin{array}{c}
a \\
b \\
c
\end{array}\right) & =a\left|e_{0}\right\rangle+b\left|e_{1}\right\rangle+c\left|e_{2}\right\rangle  \tag{6.14}\\
\left|\psi_{0}\right\rangle \otimes\left|\psi_{0}\right\rangle & =\left(\begin{array}{lll}
a^{2} & a b & a c \\
a b & b^{2} & b c \\
a c & b c & c^{2}
\end{array}\right) \tag{6.15}
\end{align*}
$$

The symmetric subspace is spanned by the basis

$$
\begin{gathered}
\left|e_{0}\right\rangle\left|e_{0}\right\rangle \\
\left|e_{1}\right\rangle\left|e_{1}\right\rangle \\
\left|e_{2}\right\rangle\left|e_{2}\right\rangle \\
\frac{1}{\sqrt{2}}\left(\left|e_{0}\right\rangle\left|e_{1}\right\rangle+\left|e_{1}\right\rangle\left|e_{0}\right\rangle\right) \\
\frac{1}{\sqrt{2}}\left(\left|e_{1}\right\rangle\left|e_{2}\right\rangle+\left|e_{2}\right\rangle\left|e_{1}\right\rangle\right) \\
\frac{1}{\sqrt{2}}\left(\left|e_{0}\right\rangle\left|e_{2}\right\rangle+\left|e_{2}\right\rangle\left|e_{0}\right\rangle\right)
\end{gathered}
$$

The anti-symmetric subspace is spanned by the $d(d-1) / 2$ vectors

$$
\begin{aligned}
& \frac{1}{\sqrt{2}}\left(\left|e_{0}\right\rangle\left|e_{1}\right\rangle-\left|e_{1}\right\rangle\left|e_{0}\right\rangle\right) \\
& \frac{1}{\sqrt{2}}\left(\left|e_{1}\right\rangle\left|e_{2}\right\rangle-\left|e_{2}\right\rangle\left|e_{1}\right\rangle\right) \\
& \frac{1}{\sqrt{2}}\left(\left|e_{0}\right\rangle\left|e_{2}\right\rangle-\left|e_{2}\right\rangle\left|e_{0}\right\rangle\right)
\end{aligned}
$$

So the product vector written in this basis is

$$
\operatorname{Sym}\left(\left|\psi_{0}\right\rangle \otimes\left|\psi_{0}\right\rangle\right)=V_{s}=\left(\begin{array}{c}
a^{2}  \tag{6.16}\\
b^{2} \\
c^{2} \\
\sqrt{2} a b \\
\sqrt{2} b c \\
\sqrt{2} a c
\end{array}\right)
$$

$V_{s}$ sits in the symmetric subspace of dimension

$$
\begin{equation*}
\frac{d(d+1)}{2}=6 \text { for } d=3 \tag{6.17}
\end{equation*}
$$

We call this vector $V_{s}$ the ETF fiducial. The action of an unusual reducible representation of the Weyl-Heisenberg group on $V_{s}$ will create an $\operatorname{ETF}_{(6,9)}$. Similarly, we can perform this construction in any dimension $d$ using a SIC fiducial to get an $\mathrm{ETF}_{1}$. The group works differently for even and odd dimensions, so we'll look at odd dimensions first.

### 6.3.1 For odd $d$

The action of the Weyl-Heisenberg group on the tensor product can be taken as

$$
\begin{align*}
& \mathbb{X}=X \otimes X  \tag{6.18}\\
& \mathbb{Z}=Z \otimes Z \tag{6.19}
\end{align*}
$$

From 2.11, where $\omega=e^{\frac{2 \pi i}{d}}$

$$
\begin{equation*}
\mathbb{Z} \mathbb{X}=\omega^{2} \mathbb{X} \mathbb{Z} \tag{6.20}
\end{equation*}
$$

For odd $d, \omega^{2}$ is a $d^{\text {th }}$ root of unity. However, for even $d, \omega^{2}$ is the $\frac{d}{2}^{\text {th }}$ root of unity and is no longer a primitive $d^{\text {th }}$ root of unity. Hence, we will discuss the case for even dimensions separately.

If we want $\mathbb{K}$ and $\mathbb{Z}$ to follow the same relation as eq 2.11 , we can choose a different set of generators for the group.

$$
\begin{align*}
& \mathbb{X}=X \otimes X  \tag{6.21}\\
& \mathbb{Z}=Z^{\frac{d+1}{2}} \otimes Z^{\frac{d+1}{2}} \tag{6.22}
\end{align*}
$$

Note that $(d+1) / 2$ is an integer as $d$ is odd. This gives us

$$
\begin{equation*}
\mathbb{Z X}=\omega \mathbb{K} \mathbb{Z} \tag{6.23}
\end{equation*}
$$

We will stick to this definition of $\mathbb{X}$ and $\mathbb{Z}$ from here on. Let us now introduce some symbols for concise writing

$$
\begin{align*}
|i, i\rangle & =|i\rangle|i\rangle  \tag{6.24}\\
|(i, j)\rangle & =\frac{1}{\sqrt{2}}(|i\rangle|j\rangle+|j\rangle|i\rangle)  \tag{6.25}\\
|[i, j]\rangle & =\frac{1}{\sqrt{2}}(|i\rangle|j\rangle-|j\rangle|i\rangle) \tag{6.26}
\end{align*}
$$

The action of the tensor product Weyl-Heisenberg group on the symmetric subspace is

$$
\begin{align*}
\mathbb{X}|i, i\rangle & =|i+1, i+1\rangle  \tag{6.27}\\
\mathbb{Z}|i, i\rangle & =\omega^{i}|i, i\rangle  \tag{6.28}\\
\mathbb{X}|(i, i+n)\rangle & =|(i+1, i+n+1)\rangle  \tag{6.29}\\
\mathbb{Z}|(i, i+n)\rangle & =\omega^{i+\left(\frac{d+1}{2}\right) n}|(i, i+n)\rangle \tag{6.30}
\end{align*}
$$

Looking at dimension 3 again, the symmetric tensor product formed by ( $X \otimes$ $X)\left(\left|\psi_{0}\right\rangle \otimes\left|\psi_{0}\right\rangle\right)$ and $(Z \otimes Z)\left(\left|\psi_{0}\right\rangle \otimes\left|\psi_{0}\right\rangle\right)$ is equivalent to the action

$$
\left(\begin{array}{c:c}
X_{3} & \mathbf{0}  \tag{6.31}\\
\hdashline \mathbf{0} & X_{3}
\end{array}\right) V_{s} \text { and }\left(\begin{array}{c:c}
Z_{3}^{2} & \mathbf{0} \\
\hdashline \mathbf{0} & \omega_{3} Z_{3}^{2}
\end{array}\right) V_{s} \quad \text { respectively. }
$$

where $V_{s}$ is the vector defined in eq. 6.16. We can see that the representation is reducible consisting of $(d+1) / 2$ blocks of $d$ dimensional matrices. Another way to come to this conclusion is looking at the repeated action of $\mathbb{X}$ on $|i, i\rangle$

$$
\begin{equation*}
|i, i\rangle \rightarrow|i+1, i+1\rangle \rightarrow \ldots|i+d-1, i+d-1\rangle \rightarrow|i, i\rangle \tag{6.32}
\end{equation*}
$$

resulting in groups of $d$ vectors. We want to rearrange the basis such that the representation becomes

$$
\mathbb{1}^{\left(\frac{d+1}{2}\right)} \otimes D_{\mathbf{p}}^{(d)}=\left(\begin{array}{lllll}
D_{\mathbf{p}} & & &  \tag{6.33}\\
& D_{\mathbf{p}} & & \\
& & \cdot & \\
& & & \\
& & & D_{\mathbf{p}}
\end{array}\right)_{\frac{d(d+1)}{2} \times \frac{d(d+1)}{2}}
$$

We choose the starting vector of the first block such that

$$
\begin{equation*}
\mathbb{Z}|i, i\rangle=|i, i\rangle \Rightarrow i=0 \tag{6.34}
\end{equation*}
$$

and act on the vector $(d-1)$ times with $\mathcal{K}$ to get the complete block. Similarly, for the next blocks

$$
\begin{equation*}
\mathbb{Z}|(i, i+n)\rangle=|(i, i+n)\rangle \Rightarrow i=d-\left(\frac{d+1}{2}\right) n \tag{6.35}
\end{equation*}
$$

For getting $(d+1) / 2$ blocks, $n$ goes from 1 to $(d-1) / 2$ in order. For example, for $d=3$, the components are choosen to align with the basis

$$
\begin{equation*}
|0,0\rangle, \quad|1,1\rangle, \quad|2,2\rangle, \quad|(1,2)\rangle, \quad|(2,0)\rangle, \quad|(0,1)\rangle \tag{6.36}
\end{equation*}
$$

The vector $V_{s}$ is redefined to be

$$
V_{s}=\left(\begin{array}{c}
a^{2}  \tag{6.37}\\
b^{2} \\
c^{2} \\
\sqrt{2} b c \\
\sqrt{2} a c \\
\sqrt{2} a b
\end{array}\right)
$$

By construction, such a change of basis creates a vector which is of the form

$$
|u\rangle=\left(\begin{array}{c}
\mathbf{x}_{1}  \tag{6.38}\\
\mathbf{x}_{2} \\
\vdots \\
\mathbf{x}_{\frac{d+1}{2}}
\end{array}\right)
$$

where $\mathbf{x}_{k} \in \mathbb{C}^{d}$ and

$$
\begin{equation*}
\left\langle x_{i} \mid x_{j}\right\rangle=\frac{2}{d+1} \delta_{i j} \tag{6.39}
\end{equation*}
$$

Chapter 6

To see that such a fiducial forms an ETF for the given representation of the Weyl-Heisenberg Group in 6.33, we prove the following theorem.

Theorem 2. For a vector

$$
|u\rangle=\left(\begin{array}{c}
\mathbf{x}_{1}  \tag{6.40}\\
\mathbf{x}_{2} \\
\vdots \\
\mathbf{x}_{m}
\end{array}\right)
$$

where $\mathbf{x}_{k} \in \mathbb{C}^{n}$, the $n^{2}$ vectors

$$
\begin{equation*}
\left|\phi_{\mathbf{p}}\right\rangle=\mathbb{1}^{(m)} \otimes D_{\mathbf{p}}^{(n)}|u\rangle \tag{6.41}
\end{equation*}
$$

form a tight frame iff

$$
\begin{equation*}
\left\langle x_{i} \mid x_{j}\right\rangle=\frac{1}{m} \delta_{i j} \tag{6.42}
\end{equation*}
$$

Proof. Construct the matrix S with vectors $\mathbb{1}^{(m)} \otimes D_{\mathbf{p}}^{(n)}|u\rangle$ as columns

$$
\mathrm{S}=\left(\begin{array}{cccc}
\left|x_{1}\right\rangle & D_{0,1}\left|x_{1}\right\rangle & \ldots & D_{n-1, n-1}\left|x_{1}\right\rangle  \tag{6.43}\\
\left|x_{2}\right\rangle & D_{0,1}\left|x_{2}\right\rangle & \ldots & D_{n-1, n-1}\left|x_{2}\right\rangle \\
\vdots & \vdots & & \vdots \\
\left|x_{m}\right\rangle & D_{0,1}\left|x_{m}\right\rangle & \ldots & D_{n-1, n-1}\left|x_{m}\right\rangle
\end{array}\right)_{m n \times n^{2}}
$$

Computing $S S^{\dagger}$

$$
\begin{align*}
S S_{i j}^{\dagger} & =\sum_{\mathbf{p}} D_{\mathbf{p}} \mathbf{x}_{i} \mathbf{x}_{j}^{\dagger} D_{\mathbf{p}}^{\dagger}  \tag{6.44}\\
& =n\left\langle x_{i} \mid x_{j}\right\rangle \mathbb{1}_{n} \tag{6.45}
\end{align*}
$$

as the Weyl-Heisenberg Group forms a unitary operator basis and hence

$$
\begin{equation*}
\sum_{\mathbf{p}} D_{\mathbf{p}} A D_{\mathbf{p}}^{-1}=n \operatorname{Tr}(A) \mathbb{1}_{n} \tag{6.46}
\end{equation*}
$$

For the vectors $\left\{\left|\phi_{i}\right\rangle\right\}_{i=1}^{n^{2}}$ to form a tight frame,

$$
\begin{align*}
S S^{\dagger} & =\left(\begin{array}{lll}
\left|\phi_{1}\right\rangle & \ldots & \left|\phi_{n^{2}}\right\rangle
\end{array}\right)\left(\begin{array}{c}
\left\langle\phi_{1}\right| \\
\vdots \\
\left\langle\phi_{n^{2}}\right|
\end{array}\right)  \tag{6.47}\\
& =\sum_{i=1}^{n^{2}}\left|\phi_{i}\right\rangle\left\langle\phi_{i}\right|  \tag{6.48}\\
& =\frac{n}{m} \mathbb{1}_{m n} \tag{6.49}
\end{align*}
$$

We can see that this is possible if and only if

$$
\begin{equation*}
\left\langle x_{i} \mid x_{j}\right\rangle=\frac{1}{m} \delta_{i j} \tag{6.50}
\end{equation*}
$$

## Generalized Parity Operator

Given that we have an ETF fiducial $|u\rangle$ constructed from a SIC (eq. 6.38), we can construct an operator $P_{\theta}$, which was originally defined by Appleby et al. (2017) [18] as the generalized parity operator and played a significant role in the paper. We will now state a theorem due to Ostrovskyi and Yakymenko (2019) [30].

Theorem 3. For the vectors $\mathbf{x}_{k}$ in eq. 6.38, we can construct the generalized parity operator from the projector $\sum_{k}\left|x_{k}\right\rangle\left\langle x_{k}\right|$ such that

$$
\begin{equation*}
\sum_{k=1}^{\frac{d+1}{2}}\left|x_{k}\right\rangle\left\langle x_{k}\right|=\frac{1}{2}\left(\mathbb{1}+P_{\theta}\right) \tag{6.51}
\end{equation*}
$$

where

$$
\begin{equation*}
P_{\theta}=\frac{1}{d} \sum_{i, j} e^{2 i i_{i, 2^{-1} j}} D_{(-i,-j)} \tag{6.52}
\end{equation*}
$$

Proof. We defined the Weyl-Heisenberg Group in the tensor product space by

$$
\begin{align*}
& \mathbb{K}=X \otimes X  \tag{6.53}\\
& \mathbb{Z}=Z^{\frac{d+1}{2}} \otimes Z^{\frac{d+1}{2}} \tag{6.54}
\end{align*}
$$

such that the displacement operators are

$$
\begin{equation*}
\tilde{D}_{i, j}=\tau^{i j} \not^{i} \mathbb{Z}^{j}=D_{i, 2^{-1} j} \otimes D_{i, 2^{-1} j} \tag{6.55}
\end{equation*}
$$

We can rewrite this as

$$
\begin{equation*}
\tilde{D}_{\mathbf{p}}=D_{H \mathbf{p}} \otimes D_{H \mathbf{p}} \tag{6.56}
\end{equation*}
$$

where

$$
H=\left(\begin{array}{cc}
1 & 0  \tag{6.57}\\
0 & 2^{-1}
\end{array}\right)
$$

As we saw earlier, the displacement operator $\tilde{D}_{\mathbf{p}}$ has an equivalent representation as

$$
\begin{equation*}
\tilde{D}_{\mathbf{p}}=\mathbb{1}^{(d)} \otimes D_{\mathbf{p}}^{(d)} \tag{6.58}
\end{equation*}
$$

or

$$
\begin{equation*}
\tilde{D}_{\mathbf{p}}=\mathbb{1}^{\left(\frac{d+1}{2}\right)} \otimes D_{\mathbf{p}}^{(d)} \tag{6.59}
\end{equation*}
$$

if we are looking at the symmetric subspace. Recalling the overlap phase factors from eq. 4.2, we see that

$$
\begin{align*}
\left\langle\psi_{0}\right|\left\langle\psi_{0}\right| \tilde{D}_{\mathbf{p}}\left|\psi_{0}\right\rangle\left|\psi_{0}\right\rangle & =\left(\left\langle\psi_{0}\right| D_{H \mathbf{p}}\left|\psi_{0}\right\rangle\right)^{2}  \tag{6.60}\\
& = \begin{cases}1 & \mathbf{p}=0 \\
\frac{1}{d+1} e^{2 i \theta_{H \mathbf{p}}} & \text { otherwise }\end{cases} \tag{6.61}
\end{align*}
$$

Normalizing $|u\rangle$ from eq. 6.38,

$$
\begin{align*}
\langle u| \tilde{D}_{\mathbf{p}}|u\rangle & =\langle u| \mathbb{1}^{\left(\frac{d+1}{2}\right)} \otimes D_{\mathbf{p}}^{(d)}|u\rangle  \tag{6.62}\\
& =\sum_{k=1}^{\frac{d+1}{2}}\left\langle x_{k}\right| D_{\mathbf{p}}\left|x_{k}\right\rangle  \tag{6.63}\\
& = \begin{cases}\frac{d+1}{2} & \mathbf{p}=0 \\
\frac{1}{2} e^{2 i \theta_{H \mathbf{p}}} & \text { otherwise }\end{cases} \tag{6.64}
\end{align*}
$$

Now, expanding the projector

$$
\begin{align*}
\sum_{k=1}^{\frac{d+1}{2}}\left|x_{k}\right\rangle\left\langle x_{k}\right| & =\frac{1}{d} \sum_{\mathbf{p}} \sum_{k}\left\langle x_{k}\right| D_{\mathbf{p}}\left|x_{k}\right\rangle D_{-\mathbf{p}}  \tag{6.65}\\
& =\frac{1}{d} \frac{d+1}{2} \mathbb{1}+\frac{1}{2 d} \sum_{\mathbf{p} \neq 0} e^{2 i \theta_{H \mathbf{p}}} D_{-\mathbf{p}}  \tag{6.66}\\
& =\frac{1}{d} \frac{d+1}{2} \mathbb{1}-\frac{1}{2 d} \mathbb{1}+\frac{1}{2 d} \sum_{\mathbf{p}} e^{2 i \theta_{H \mathbf{p}}} D_{-\mathbf{p}}  \tag{6.67}\\
& =\frac{1}{2}\left(\mathbb{1}+P_{\theta}\right) \tag{6.68}
\end{align*}
$$

It follows that

$$
\begin{equation*}
P_{\theta}=\frac{1}{d} \sum_{\mathbf{p}} e^{2 i \theta_{H \mathbf{p}}} D_{-\mathbf{p}} \tag{6.69}
\end{equation*}
$$

### 6.3.2 For even $d$

We will now take a modified approach to create these ETFs in even dimensions. For even dimensions, the representation is given by Andersson and Dumitru (2019) [31] and Ostrovskyi and Yakymenko (2019) [30]. For example, for $d=4$ the operator consists of $(d+1)$ blocks of $d / 2$ dimensional displacement operators with a phase factor.

$$
\tilde{D}_{i, j}=(-1)^{i j}\left(\begin{array}{ccccc}
D_{i, j}^{(d / 2)} & & & & \\
& \omega^{i} D_{i, j}^{(d / 2)} & & & \\
& & \omega^{j} D_{i, j}^{(d / 2)} & & \\
& & & \omega^{i+j} D_{i, j}^{(d / 2)} & \\
& & & & D_{i, j}^{(d / 2)}
\end{array}\right)_{\frac{d(d+1)}{2} \times \frac{d(d+1)}{2}}
$$

### 6.3.3 $\operatorname{ETF}_{(10,16)}$ from SIC in $d=4$

The case of even dimensions is a lot more complicated than in odd dimensions. For this, we take an example of $d=4$ and explain the construction on the basis of the calculations. We want to construct an $\mathrm{ETF}_{1}$ given a SIC fiducial in dimension 4. We will use the exact solution for the fiducial [24]

$$
\left|\psi_{0}\right\rangle=\left(\begin{array}{c}
a_{1} \\
a_{2} \\
a_{3} \\
a_{4}
\end{array}\right)=\frac{1}{c_{1}}\left(\begin{array}{c}
8 \\
-4+4 \sqrt{2}+c_{2}+i\left(c_{3} \sqrt{1+\sqrt{5}}-4\right) \\
i(8 \sqrt{2}-8) \\
4-4 \sqrt{2}+c_{2}+i\left(c_{3} \sqrt{1+\sqrt{5}}+4\right)
\end{array}\right)
$$

where

$$
\begin{align*}
& c_{1}=8 \sqrt{(2-\sqrt{2})(5+\sqrt{5})}  \tag{6.70}\\
& c_{2}=\sqrt{1+\sqrt{5}}(\sqrt{2}+\sqrt{10})  \tag{6.71}\\
& c_{3}=(-2+\sqrt{2}-2 \sqrt{5}+\sqrt{10}) \tag{6.72}
\end{align*}
$$

After taking the tensor product, $\left|\psi_{0}\right\rangle \otimes\left|\psi_{0}\right\rangle$, we get 16 components in the normal basis. To get the symmetric subspace and to choose a basis such that we can use the representation of the Weyl-Heisenberg group as given in eq. 6.3.2, we select the eigenbasis following Section 4.2 in Ostrovskyi and Yakymenko (2019) [30],

$$
\begin{aligned}
& |0,0\rangle+|2,2\rangle \\
& \begin{array}{l}
|1,1\rangle+|3,3\rangle \\
\ldots
\end{array} \\
& |0,0\rangle-|2,2\rangle \\
& -i(|1,1\rangle-|3,3\rangle) \\
& |(0,1)\rangle+|(2,3)\rangle \\
& |(1,2)\rangle+|(3,0)\rangle \\
& |(0,1)\rangle-|(2,3)\rangle \\
& -i(|(1,2)\rangle-|(3,0)\rangle) \\
& |(1,3)\rangle \\
& |(0,2)\rangle
\end{aligned}
$$

In this basis, we get the vector which forms our ETF fiducial.

$$
\left(\begin{array}{c}
\frac{1}{2}\left(a_{1}^{2}+a_{3}^{2}\right)  \tag{6.73}\\
\frac{1}{2}\left(a_{2}^{2}+a_{4}^{2}\right) \\
\frac{1}{2}\left(a_{1}^{2}-a_{3}^{2}\right) \\
-\frac{i}{2}\left(a_{2}^{2}-a_{4}^{2}\right) \\
\frac{1}{\sqrt{2}}\left(a_{1} a_{2}+a_{3} a_{4}\right) \\
\frac{1}{\sqrt{2}}\left(a_{1} a_{4}+a_{2} a_{3}\right) \\
\frac{1}{\sqrt{2}}\left(a_{1} a_{2}-a_{3} a_{4}\right) \\
-\frac{i}{\sqrt{2}}\left(a_{2} a_{3}-a_{1} a_{4}\right) \\
a_{2} a_{4} \\
a_{1} a_{3}
\end{array}\right)
$$

Now, acting on this vector with the matrix

$$
\tilde{D}_{i, j}=(-1)^{i j}\left(\begin{array}{ccccc}
D_{i, j}^{(2)} & & & &  \tag{6.74}\\
& \omega^{i} D_{i, j}^{(2)} & & & \\
& & \omega^{j} D_{i, j}^{(2)} & & \\
& & & \omega^{i+j} D_{i, j}^{(2)} & \\
& & & & D_{i, j}^{(2)}
\end{array}\right)_{10 \times 10}
$$

where $0 \leq i, j \leq 3$, we get the 16 vectors which form the $\operatorname{ETF}_{(10,16)}$. Here $\omega=e^{\frac{2 \pi i}{4}}$. The matrix we get is complicated but it can easily be checked that it forms an ETF. We can see that the rows are orthogonal to each other. Also, the 16 unit vectors which sit in $d(d+1) / 2=10$ follow

$$
\begin{equation*}
\left|\left\langle\Psi_{i} \mid \Psi_{j}\right\rangle\right|=\frac{1}{5} \quad \text { where } \quad i \neq j \tag{6.75}
\end{equation*}
$$

### 6.4 Constructing $\operatorname{ETF}_{\left(\frac{d(d-1)}{2}, d^{2}\right)}$ from $\operatorname{ETF}_{\left(\frac{d(d+1)}{2}, d^{2}\right)}$

We now know how to create an $\mathrm{ETF}_{1}$ given a SIC fiducial in dimension $d$. We move on to the next step towards our goal, that is to create a Naimark Complement for our $\mathrm{ETF}_{1}$. As

$$
\begin{equation*}
d^{2}-\frac{d(d+1)}{2}=\frac{d(d-1)}{2} \tag{6.76}
\end{equation*}
$$

the ETF can be constructed easily using the Naimark Theorem. Again, for simplicity, we want to call the $\operatorname{ETF}_{\left(\frac{d(d-1)}{2}, d^{2}\right)}$ as $\mathrm{ETF}_{2}$. A proof of the existence of an $\mathrm{ETF}_{2}$, given a SIC in dimension $d$ has also been given by Appleby et al. (2019) [32], which works at the level of Gram matrices.

### 6.4.1 For odd $d$

Given that we have an $\mathrm{ETF}_{1}$, the ETF fiducial is given by

$$
|u\rangle=\left(\begin{array}{c}
\mathbf{x}_{1}  \tag{6.77}\\
\mathbf{x}_{2} \\
\vdots \\
\mathbf{x}_{\frac{d+1}{2}}
\end{array}\right)
$$

In order to find the Naimark Complement, we use Theorem 2 to find an $\mathrm{ETF}_{2}$ fiducial

$$
|v\rangle=\left(\begin{array}{c}
\mathbf{y}_{1}  \tag{6.78}\\
\mathbf{y}_{2} \\
\vdots \\
\mathbf{y}_{\frac{d-1}{2}}
\end{array}\right)
$$

such that $\mathbf{y}_{k} \in \mathbb{C}^{d}$,

$$
\begin{equation*}
\left\langle y_{i} \mid y_{j}\right\rangle=\frac{2}{d-1} \delta_{i j} \tag{6.79}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\langle x_{i} \mid y_{j}\right\rangle=0 \tag{6.80}
\end{equation*}
$$

Then, the ETF is formed by the action of the group

$$
\begin{equation*}
\mathbb{1}^{\left(\frac{d-1}{2}\right)} \otimes D_{\mathbf{p}}^{(d)} \tag{6.81}
\end{equation*}
$$

on $|v\rangle$. As the process of finding the complement involves completing the basis of the vectors $\mathbf{x}_{k}$, there is no unique way of finding this complement.

Theorem 4. The $E T F_{2}$ constructed using the fiducial $|v\rangle$ has squared overlap phases $\pm e^{2 i \theta_{\mathrm{p}}}$.

Proof. As ETF ${ }_{1}$ created using $|u\rangle$ as the fiducial has squared overlap phases by construction (eq. 6.62), the proof follows from eq. 6.4.

### 6.4.2 For even $d$

We will again look at the case of $d=4$ for even dimensions for constructing the Naimark Complement. As we have already constructed the $\operatorname{ETF}_{(10,16)}$ given in Section 6.3.3, we can work with it to show the construction of ETF ${ }_{2}$. Here, the ETF can also be called $\operatorname{ETF}_{(6,16)}$. Let the fiducial of its Naimark complement be given by a vector whose components are unknown.

$$
\mathrm{ETF}_{2} \text { fiducial }=\left(\begin{array}{c}
x_{1}  \tag{6.82}\\
x_{2} \\
x_{3} \\
x_{4} \\
x_{5} \\
x_{6}
\end{array}\right)
$$

We act on this vector with

$$
\tilde{D}_{i, j}=(-1)^{i j}\left(\begin{array}{ccc}
\omega^{i} D_{i, j}^{(2)} & &  \tag{6.83}\\
& \omega^{j} D_{i, j}^{(2)} & \\
& & \omega^{i+j} D_{i, j}^{(2)}
\end{array}\right)_{6 \times 6}
$$

where $0 \leq i, j \leq 3$, and hence get 16 vectors. We chose this representation of the displacement operators in 6 dimensions as it completes the block diagonal matrix of eq. 6.74. Putting these vectors as the columns of a matrix of size $(6 \times 16)$, we create what should be an $\operatorname{ETF}_{(6,16)}$. As these ETFs together should form a unitary matrix, $U U^{\dagger}=\mathbb{1}$.

$$
U=\left(\begin{array}{ccc}
\ldots & \operatorname{ETF}_{(10,16)} & \ldots  \tag{6.84}\\
& & \\
\ldots & \operatorname{ETF}_{(6,16)} & \cdots
\end{array}\right)_{16 \times 16}
$$

Then, the displacement operator which acts on the unitary matrix $U$ is given by

$$
\tilde{D}_{i, j}^{(16)}=(-1)^{i j}\left(\begin{array}{cccccccc}
D_{i, j}^{(2)} & & & & & & & \\
& \omega^{i} D_{i, j}^{(2)} & & & & & & \\
& & \omega^{j} D_{i, j}^{(2)} & & & & \\
& & & \omega^{i+j} D_{i, j}^{(2)} & & & & \\
& & & & D_{i, j}^{(2)} & & & \\
& & & & & \omega^{i} D_{i, j}^{(2)} & & \\
& & & & & & \omega^{j} D_{i, j}^{(2)} & \\
& & & & & & & \omega^{i+j} D_{i, j}^{(2)}
\end{array}\right)
$$

Now, putting $U U^{\dagger}=\mathbb{1}$, we get 6 unique conditions to determine the vector. Solving for the 6 unknown components in our initial $\mathrm{ETF}_{2}$ fiducial, we get the vector as

$$
|\phi\rangle_{6}=\left(\begin{array}{c}
a_{1}  \tag{6.85}\\
a_{2} \\
a_{3} e^{i \theta_{1}} \\
a_{4} e^{i \theta_{1}} \\
a_{5} e^{i \theta_{2}} \\
a_{6} e^{i \theta_{2}}
\end{array}\right)
$$

where

$$
\begin{align*}
a_{1} & =\frac{1}{4} \sqrt{\frac{1}{2}(1+\sqrt{5})}  \tag{6.86}\\
a_{3} & =\frac{1}{4}  \tag{6.87}\\
a_{2} & =\frac{1}{2(1+\sqrt{5})}  \tag{6.88}\\
a_{4} & =\frac{1}{8}(1-\sqrt{5}-i \sqrt{2(-1+\sqrt{5})})  \tag{6.89}\\
a_{5} & =\frac{1}{8} \sqrt{4+\sqrt{2(1+\sqrt{5})}-\sqrt{10(1+\sqrt{5})}} \\
a_{6} & =\frac{-i}{16}(1+\sqrt{5}+\sqrt{2(1+\sqrt{5})}) \sqrt{4+\sqrt{2(1+\sqrt{5})}-\sqrt{10(1+\sqrt{5})}}
\end{align*}
$$

We can see that the vector is not uniquely determined and is unknown in up to 2 phase factors. It is interesting to note that all the constraints on the unknowns happen in pairs, due to the nature of the Weyl-Heisenberg group representation in even dimensions where for $d=4$ we have 3 copies of the 2 -dimensional displacement operators $D_{\mathbf{p}}^{(2)}$.

### 6.5 Constructing $\mathbf{E T F}_{2}$ from SIC $_{\left(d(d-2), d^{2}(d-2)^{2}\right)}$

We now know how to create an $\mathrm{ETF}_{1}$ and $\mathrm{ETF}_{2}$ given only a SIC fiducial in dimension $d$. Interestingly, there is another way to create the $\mathrm{ETF}_{2}$ in dimension $d(d-1) / 2$ from an aligned SIC in dimension $d(d-2)$. With this, we finally have a way to connect the SIC in $d$ and one in $d(d-2)$.

$$
\mathrm{SIC}_{d} \rightarrow \mathrm{ETF}_{1} \rightarrow \mathrm{ETF}_{2} \rightarrow \mathrm{SIC}_{d(d-2)}
$$

We would like to know if an $\mathrm{ETF}_{2}$ can be constructed from a $\mathrm{SIC}_{d(d-2)}$. Using the equiangular condition (eq. 3.15) for an $\operatorname{ETF}_{\left(\frac{d(d-1)}{2}, d^{2}\right)}$,

$$
\begin{equation*}
\left|\left\langle\psi_{I} \mid \psi_{J}\right\rangle\right|^{2}=\frac{1}{d(d-2)+1} \tag{6.90}
\end{equation*}
$$

which says that the ETF can sit inside the $\operatorname{SIC}_{d(d-2)}$.
The connection between $\mathrm{ETF}_{2}$ and the $\mathrm{SIC}_{d(d-2)}$ comes from the special nature of dimension $d(d-2)$ discussed in Chapter 4. We will use the discussion on overlap phases in our discussion and later in our calculations.

Let the overlap phases for a $\mathrm{SIC}_{d}$ be given by $e^{i \theta_{\mathrm{p}}}$. Due to the tensor product nature in the construction, the overlap phases for ETF ETe $_{1}$ are $e^{2 i \theta_{\mathrm{p}}}$. Using eq. 6.4, we find that the overlap phase factors for $\mathrm{ETF}_{2}$ are given by

$$
\begin{equation*}
\left\langle v_{I} \mid v_{J}\right\rangle=-e^{2 i \theta_{\mathrm{p}}} \tag{6.91}
\end{equation*}
$$

Referring to eq. 4.4, the overlap phases of the $\mathrm{SIC}_{d(d-2)}$ are also given by

$$
\begin{equation*}
e^{i \Theta_{\mathbf{p}^{\prime}}}= \pm e^{2 i \theta_{\mathbf{p}}} \tag{6.92}
\end{equation*}
$$

for selected values of $\mathbf{p}^{\prime}$. With this relation in mind, we can try to construct an $\mathrm{ETF}_{2}$ from a $\mathrm{SIC}_{d(d-2)}$ such that it is a Naimark Complement to the $\mathrm{ETF}_{1}$ we created from $\mathrm{SIC}_{d}$. We shall look at the exact steps to do this below.

### 6.5.1 For odd $d$

For a SIC in dimension $d(d-2)$ where $d$ is odd, recall that there exists an extra symmetry (eq. 5.13) [18]

$$
\begin{equation*}
U_{P}^{(d-2)} \otimes \mathbb{1}^{(d)} \tag{6.93}
\end{equation*}
$$

where P is the parity operator given by

$$
P=\left(\begin{array}{cc}
-1 & 0  \tag{6.94}\\
0 & -1
\end{array}\right) ; \quad U_{P}^{2}=\mathbb{1}
$$

As the spectrum of the unitary was given by $((d+1) / 2,(d-1) / 2)$, diagonalizing $U_{P}$ and making this change of basis for the $\mathrm{SIC}_{d(d-2)}$, we get a vector containing only $d(d-1) / 2$ non-zero components. We would like to check if this vector can act as a fiducial for $\mathrm{ETF}_{2}$. Removing the zero components and acting on it with $\mathbb{1}^{\left(\frac{d-1}{2}\right)} \otimes D_{\mathbf{p}}^{(d)}$, we get a $\frac{d(d-1)}{2} \times d^{2}$ matrix. For this to be our $\mathrm{ETF}_{2}$, we need it to be the Naimark complement of the ETF ${ }_{1}$ created from SIC $_{d}$.

$$
\left(\begin{array}{cccc}
\left|u_{1}\right\rangle & \left|u_{2}\right\rangle & \ldots & \left|u_{d^{2}}\right\rangle  \tag{6.95}\\
\hline\left|v_{1}^{\prime}\right\rangle & \left|v_{2}^{\prime}\right\rangle & \ldots & \left|v_{d^{2}}^{\prime}\right\rangle
\end{array}\right)_{d^{2} \times d^{2}}
$$

such that it follows

$$
\begin{equation*}
\left\langle u_{i} \mid u_{j}\right\rangle+\left\langle v_{i} \mid v_{j}\right\rangle=0 \quad i \neq j \tag{6.96}
\end{equation*}
$$

We check the above procedure for a few aligned SICs by acting on $\left|v_{1}^{\prime}\right\rangle$ with $\mathbb{1}^{\left(\frac{d-1}{2}\right)} \otimes D_{C \mathbf{p}}^{(d)}$, where $C$ is determined individually in order to align the vectors correctly. A few results are given below.

| d | $\mathrm{d}(\mathrm{d}-2)$ | C |
| :---: | :---: | :---: |
| 5 a | 15 d | $\left(\begin{array}{cc}-1 & 0 \\ 0 & -1\end{array}\right)$ |
| 7 a | 35 i | $\left(\begin{array}{cc}1 & 4 \\ 4 & 3\end{array}\right)$ |
| 9 a | 63 b | $\left(\begin{array}{cc}0 & 1 \\ -1 & 0\end{array}\right)$ |

Table 6.1: Aligning SICs in dimensions d and d(d-2)

The above alignment does create for us an ETF which is the Naimark Complement of ETF $_{1}$ in these dimensions. The goal now would be to create a SIC in $d(d-2)$ starting only from the SIC in dimension $d$.

Consider the Naimark Complement given in eq. 6.78 which was created solely from the $\mathrm{SIC}_{d}$ by taking the Naimark Complement of the fiducial of ETF ${ }_{1}$. We can now lift it up to dimension $d(d-2)$ using the inverse of the procedure we used above.

$$
|v\rangle=\left(\begin{array}{c}
\mathbf{y}_{1}  \tag{6.97}\\
\mathbf{y}_{2} \\
\vdots \\
\mathbf{y}_{\frac{d-1}{2}}
\end{array}\right)
$$

Then, the SIC in $d(d-2)$ would have another $d(d-3) / 2$ components as 0 .

$$
\text { Let }\left|\Psi^{\prime}\right\rangle=\frac{1}{\sqrt{\frac{d-1}{2}}}\left(\begin{array}{c}
\mathbf{y}_{1}  \tag{6.98}\\
\vdots \\
\mathbf{y}_{\frac{d-1}{2}} \\
\mathbf{0}_{1} \\
\vdots \\
\mathbf{0}_{\frac{d-3}{2}}
\end{array}\right)
$$

Let $T$ be a matrix which diagonalizes $U_{P}$. We can make the change of basis in reverse to get a vector with $d(d-2)$ components, which can be non-zero.

$$
\begin{equation*}
|\Psi\rangle=\left(T^{(d-2)} \otimes \mathbb{1}^{(d)}\right)^{\dagger}\left|\Psi^{\prime}\right\rangle \tag{6.99}
\end{equation*}
$$

Then this new vector $|\Psi\rangle$ should be our SIC fiducial in dimension $d(d-2)$.
Theorem 5. For a SIC fiducial vector $|\Psi\rangle_{d(d-2)}$ formed using an $E T F_{2}$ fiducial $|v\rangle$,

$$
\begin{equation*}
(d-1)\langle\Psi|\left(\mathbb{1}^{(d-2)} \otimes D_{\mathbf{p}}^{(d)}\right)|\Psi\rangle= \pm e^{2 i \theta_{\mathbf{p}}} \tag{6.100}
\end{equation*}
$$

Proof. We know from Theorem 4 that the $\mathrm{ETF}_{2}$ fiducial $|v\rangle$ has squared overlap phases

$$
\begin{align*}
\pm e^{2 i \theta_{\mathbf{p}}} & =\langle v|\left(\mathbb{1}^{\left(\frac{d-1}{2}\right)} \otimes D_{\mathbf{p}}^{(d)}\right)|v\rangle  \tag{6.101}\\
& =(d-1)\left\langle\Psi^{\prime}\right|\left(\mathbb{1}^{(d-2)} \otimes D_{\mathbf{p}}^{(d)}\right)\left|\Psi^{\prime}\right\rangle  \tag{6.102}\\
& =(d-1)\langle\Psi|\left(T^{(d-2)} \otimes \mathbb{1}^{(d)}\right)^{\dagger}\left(\mathbb{1}^{(d-2)} \otimes D_{\mathbf{p}}^{(d)}\right)\left(T^{(d-2)} \otimes \mathbb{1}^{(d)}\right)|\Psi\rangle  \tag{6.103}\\
& =(d-1)\langle\Psi|\left(\mathbb{1}^{(d-2)} \otimes D_{\mathbf{p}}^{(d)}\right)|\Psi\rangle \tag{6.104}
\end{align*}
$$

Theorem 6. For a SIC fiducial vector $|\Psi\rangle_{d(d-2)}$ formed using an ETF $F_{2}$ fiducial $|v\rangle$,

$$
\begin{equation*}
(d-1)\langle\Psi|\left(D_{\mathbf{p}}^{(d-2)} \otimes \mathbb{1}^{(d)}\right)|\Psi\rangle=1 \tag{6.105}
\end{equation*}
$$

Proof.

$$
\begin{align*}
& \langle\Psi|\left(D_{\mathbf{p}}^{(d-2)} \otimes \mathbb{1}^{(d)}\right)|\Psi\rangle  \tag{6.106}\\
& =\left\langle\Psi^{\prime}\right|\left(T^{(d-2)} \otimes \mathbb{1}^{(d)}\right)\left(D_{\mathbf{p}}^{(d-2)} \otimes \mathbb{1}^{(d)}\right)\left(T^{(d-2)} \otimes \mathbb{1}^{(d)}\right)^{\dagger}\left|\Psi^{\prime}\right\rangle  \tag{6.107}\\
& =\left\langle\Psi^{\prime}\right|\left(T D_{\mathbf{p}} T^{\dagger} \otimes \mathbb{1}^{\prime}\right)\left|\Psi^{\prime}\right\rangle  \tag{6.108}\\
& =\left\langle\Psi^{\prime}\right|\left(T D_{\mathbf{p}} T^{\dagger} \otimes R^{\dagger} R\right)\left|\Psi^{\prime}\right\rangle \tag{6.109}
\end{align*}
$$

where $R$ is the matrix which diagonalizes $P_{\theta}$. As the vectors $\mathbf{y}_{k}$ are eigenvectors of $\mathrm{P}_{\theta}$, we get

$$
\left|\Psi^{\prime \prime}\right\rangle=\left(\mathbb{1}^{(d-2)} \otimes R^{(d)}\right)\left|\Psi^{\prime}\right\rangle=\frac{1}{\sqrt{\frac{d-1}{2}}}\left(\begin{array}{c}
\mathbf{e}_{1}  \tag{6.110}\\
\vdots \\
\mathbf{e}_{\frac{d-1}{2}}^{2} \\
\mathbf{0}_{1} \\
\vdots \\
\mathbf{0}_{\frac{d-3}{2}}
\end{array}\right)
$$

$$
\begin{equation*}
\langle\Psi|\left(D_{\mathbf{p}}^{(d-2)} \otimes \mathbb{1}^{(d)}\right)|\Psi\rangle=\left\langle\Psi^{\prime \prime}\right|\left(T D_{\mathbf{p}} T^{\dagger} \otimes \mathbb{1}\right)\left|\Psi^{\prime \prime}\right\rangle \tag{6.111}
\end{equation*}
$$

The matrix $T^{(d-2)}$ has the form

$$
\left(\begin{array}{ccccccc}
1 & & & & & &  \tag{6.112}\\
& \frac{1}{\sqrt{2}} & & & & & \frac{1}{\sqrt{2}} \\
& & \frac{1}{\sqrt{2}} & & & \frac{1}{\sqrt{2}} & \\
& & & \ddots & . . & & \\
& & & . & \ddots & & \\
& & \frac{1}{\sqrt{2}} & & & \frac{-1}{\sqrt{2}} & \\
& \frac{1}{\sqrt{2}} & & & & & \frac{-1}{\sqrt{2}}
\end{array}\right)
$$

Now, as $T^{\dagger}=T$,

$$
\begin{gather*}
(T \otimes \mathbb{1})\left|\Psi^{\prime \prime}\right\rangle=(T \otimes \mathbb{1}) \frac{1}{\sqrt{\frac{d-1}{2}}}\left(\begin{array}{c}
\mathbf{e}_{1} \\
\vdots \\
\mathbf{e}_{\frac{d-1}{2}} \\
0_{1} \\
\vdots \\
0 \frac{d-3}{2}
\end{array}\right)=\frac{1}{\sqrt{\frac{d-1}{2}}}\left(\begin{array}{c}
\mathbf{e}_{1} \\
\mathbf{e}_{2} / \sqrt{2} \\
\mathbf{e}_{3} / \sqrt{2} \\
\vdots \\
\mathbf{e}_{\frac{d-1}{2}} / \sqrt{2} \\
\mathbf{e}_{\frac{d-1}{2}} / \sqrt{2} \\
\vdots \\
\mathbf{e}_{3} / \sqrt{2} \\
\mathbf{e}_{2} / \sqrt{2}
\end{array}\right)  \tag{6.113}\\
|\tilde{\Psi}\rangle=\frac{1}{\sqrt{d-1}}\left(\begin{array}{c}
\sqrt{2} \mathbf{e}_{1} \\
\mathbf{e}_{2} \\
\vdots \\
\mathbf{e}_{\frac{d-1}{2}} \\
\mathbf{e}_{\frac{d-1}{2}} \\
\vdots \\
\mathbf{e}_{2}
\end{array}\right) \tag{6.114}
\end{gather*}
$$

where

$$
\begin{equation*}
\langle\Psi|\left(D_{\mathbf{p}}^{(d-2)} \otimes \mathbb{1}^{(d)}\right)|\Psi\rangle=\langle\tilde{\Psi}|\left(D_{\mathbf{p}}^{(d-2)} \otimes \mathbb{1}^{(d)}\right)|\tilde{\Psi}\rangle \tag{6.115}
\end{equation*}
$$

So now,

$$
\begin{align*}
& \left(Z^{j} \otimes \mathbb{1}\right)|\tilde{\Psi}\rangle=\frac{1}{\sqrt{d-1}}\left(\begin{array}{c}
\sqrt{2} \mathbf{e}_{1} \\
\omega^{j} \mathbf{e}_{2} \\
\vdots \\
\omega^{(d-3) j / 2} \mathbf{e}_{\frac{d-1}{2}} \\
\omega^{(d-1) j / 2} \mathbf{e}_{\frac{d-1}{2}}^{2} \\
\vdots \\
\omega^{(d-3) j} \mathbf{e}_{2}
\end{array}\right)  \tag{6.116}\\
& \langle\tilde{\Psi}|\left(Z^{j} \otimes \mathbb{1}\right)|\tilde{\Psi}\rangle=\frac{1}{d-1}\left(2+\omega^{j}+\omega^{2 j}+\cdots \omega^{(d-3) j}\right)=\frac{1}{d-1} \tag{6.117}
\end{align*}
$$

Here,

$$
\begin{gather*}
\omega=e^{\frac{2 \pi i}{(d-2)}}  \tag{6.118}\\
\Rightarrow(d-1)\langle\tilde{\Psi}|\left(Z^{j} \otimes \mathbb{1}\right)|\tilde{\Psi}\rangle=1 \tag{6.119}
\end{gather*}
$$

Also,

$$
\begin{align*}
(X \otimes \mathbb{1})|\tilde{\Psi}\rangle=\frac{1}{\sqrt{d-1}}\left(\begin{array}{c}
\mathbf{e}_{2} \\
\sqrt{2} \mathbf{e}_{1} \\
\mathbf{e}_{2} \\
\vdots \\
\mathbf{e}_{\frac{d-1}{2}} \\
\mathbf{e}_{\frac{d-1}{2}} \\
\vdots \\
\mathbf{e}_{3}
\end{array}\right)  \tag{6.120}\\
\left(X^{2} \otimes \mathbb{1}\right)|\tilde{\Psi}\rangle=\frac{1}{\sqrt{d-1}}\left(\begin{array}{c}
\mathbf{e}_{3} \\
\mathbf{e}_{2} \\
\sqrt{2} \mathbf{e}_{1} \\
\mathbf{e}_{2} \\
\vdots \\
\mathbf{e}_{\frac{d-1}{2}}^{2} \\
\mathbf{e}_{\frac{d-1}{2}} \\
\vdots \\
\mathbf{e}_{4}
\end{array}\right) \tag{6.121}
\end{align*}
$$

and so on. Thus,

$$
\begin{equation*}
(d-1)\langle\tilde{\Psi}|\left(X^{i} \otimes \mathbb{1}\right)|\tilde{\Psi}\rangle=1 \tag{6.122}
\end{equation*}
$$

Acting on the vector with $\left(X^{i} \otimes \mathbb{1}\right)$ rotates the elements $\mathbf{e}_{k}, i$ times. Looking specifically at $D_{(1, j)}$

$$
\begin{equation*}
(d-1)\langle\tilde{\Psi}|\left(\tau^{j} X Z^{j} \otimes \mathbb{1}\right)|\tilde{\Psi}\rangle=\omega^{(d-3) j / 2} \tau^{j}=1 \tag{6.123}
\end{equation*}
$$

and at $D_{(2, j)}$

$$
\begin{equation*}
(d-1)\langle\tilde{\Psi}|\left(\tau^{2 j} X^{2} Z^{j} \otimes \mathbb{1}\right)|\tilde{\Psi}\rangle=\omega^{(d-3) j} \tau^{2 j}=1 \tag{6.124}
\end{equation*}
$$

So, for a general $D_{(i, j)}^{(d-2)}$,

$$
\begin{align*}
& \text { If } i \text { is odd }:(d-1)\langle\tilde{\Psi}|\left(D_{(i, j)} \otimes \mathbb{1}\right)|\tilde{\Psi}\rangle=\omega^{(d-2-i) j / 2} \tau^{i j}=1  \tag{6.125}\\
& \text { If } i \text { is even }:(d-1)\langle\tilde{\Psi}|\left(D_{(i, j)} \otimes \mathbb{1}\right)|\tilde{\Psi}\rangle=\omega^{\left(d-2-\frac{i}{2}\right) j} \tau^{i j}=1 \tag{6.126}
\end{align*}
$$

Hence,

$$
\begin{equation*}
(d-1)\langle\Psi|\left(D_{\mathbf{p}}^{(d-2)} \otimes \mathbb{1}^{(d)}\right)|\Psi\rangle=1 \tag{6.128}
\end{equation*}
$$

## Chapter 7

## Conclusion

In this thesis, we have provided the mathematical background necessary for understanding the basics of SIC-POVMs. A special focus was put on the WeylHeisenberg Group and its representations, which turned out to be an integral part of the thesis. We then explored the connection between SICs in dimensions $d$ and $d(d-2)$, laying the groundwork for the motivation behind looking specifically at dimensions of these forms.

Further in the work, we provide the method for creating a special kind of Equiangular Tight Frame from an existing SIC, which we call an ETF ${ }_{1}$. This is done using the construction given by Renes et al. (2004) [12]. We further create its Naimark complement, which we call an $\mathrm{ETF}_{2}$, using the Naimark extension theorem. We then make the connection from $\mathrm{ETF}_{2}$ to a SIC in the higher dimension of $d(d-2)$.

Throughout the work, we have divided the results into odd and even dimensions. Though we focus mainly on the odd dimensions as they work out to be relatively simpler, we looked at the case of $d=4$ and calculated the Naimark complement exactly up to 2 phase factors.

Finally, we give the construction of the generalized parity operator and settle an open question which arises from the two types of ETFs which can be embedded into the higher dimensional SIC. For now, the method to find a unique $\mathrm{ETF}_{2}$, Naimark complement to ETF ${ }_{1}$, which corresponds to the SIC in the higher dimension remains to be found, though the argument is supported by our observation that there is an ETF embedded in the higher dimensional SIC which acts
as a Naimark complement to $\mathrm{ETF}_{1}$.

$$
\begin{equation*}
\underset{\left(d, d^{2}\right)}{\mathrm{SIC}_{1}} \longrightarrow \underset{\left(\frac{d(d+1)}{2}, d^{2}\right)}{\mathrm{ETF}_{1}} \longrightarrow \underset{\left(\frac{d(d-1)}{2}, d^{2}\right)}{\mathrm{ETF}_{2}} \rightarrow \rightarrow \underset{\left(d(d-2), d^{2}(d-2)^{2}\right)}{\mathrm{SIC}_{2}} \tag{7.1}
\end{equation*}
$$

A logical next step could be to find a connection between an ETF 2 and the $\mathrm{SIC}_{2}$ for even dimensions. It is our hope that given a method to distinguish between the different solutions of the Naimark extension theorem, one can construct a $\mathrm{SIC}_{2}$ in the higher dimension given only $\mathrm{SIC}_{1}$.

## Appendix A

## Mathematica Codes

The calculations in this thesis are done using Wolfram Mathematica 12.0 Student Edition. We give the most used code snippets below -

## Function to generate the Displacement Operators

```
DijFunc[i_, j_, d_] :=
    (Tau = -Exp[I*\[Pi]/d];
    Dij = ConstantArray[0, {d, d}];
    For[r = 0, r < d, r++,
        For[s = 0, s < d, s++,
            Dij[[r + 1, s + 1]] = Tau^(i*j + 2*s*j)*KroneckerDelta[r,
    Mod[s + i, d]];
        ]
    ];
    Dij)
```


## Creating the $\operatorname{ETF}_{\left(\frac{d(d+1)}{2}, d^{2}\right)}$ fiducial given a $\operatorname{SIC}_{\left(d, d^{2}\right)}$ when $d$ is odd

```
(*sic_fiducial is the sic fiducial given in dimension d*)
sic_tensor = KroneckerProduct[sic_fiducial, sic_fiducial];
(*Multiplying non-diagonal elements with sqrt(2)*)
For [k = 1, k< n2 + 1, k++,
    For [m = 1,m<n2 + 1, m++,
        If[k != m,
        sic_tensor [[k, m]] = sic_tensor[[k, m]]*Sqrt[2]]
    ]
]
(*Choosing the vectors in order of the eigenbasis*)
etf_fiducial={};
For [n = 0, n< (n2 + 1)/2, n++,
    For[l = 0, l < n2, l++,
    etf_fiducial = AppendTo[etf_fiducial,
            sic_tensor [[Mod [(-n*(n2 + 1)/2) + l, n2] + 1,
                Mod[(-n*(n2 + 1)/2) + n + l, n2] + 1]]]
    ]
]
```


## Creating the Clifford Group Unitaries

We have to input the dimension $d$, as well as $\alpha, \beta, \gamma$ and $\delta$.

```
If[Mod[d, 2] == 0, db = 2*d, db = d];
F = Mod[{{\[Alpha], \[Beta]}, {\[Gamma], \[Delta]}}, db];
Tau = - Exp[I*\[Pi]/d];
checkcondition = CoprimeQ[\[Beta], db];
If[\[Beta] != 0 && checkcondition == True, switch = 0, switch =
        1];
If[checkcondition == True, \[Beta]I =
    PowerMod[\[Beta], -1, db], \[Beta]I =
    Null]; (*\[Beta]I is \[Beta] inverse*)
ket[i_] := KroneckerProduct[UnitVector[d, i + 1], {1}]
bra[i_] := KroneckerProduct[UnitVector[d, i + 1], {1}] //
        Transpose
If[switch == 0,
    (V = Table[
            Tau^Mod[((\[Beta]I)*(F[[1, 1]]*(s^2) - 2*r*s + F[[2, 2]]*(r
        *2))),
            db] ket[r].bra[s], {r, 0, d - 1}, {s, 0, d - 1}]/Sqrt[d
        ];
    UF = Total[Total[V]];
        ),
    (
    F1 = {{0, -1}, {1, 0}};
    F2 = {{\[Gamma], \[Delta]}, {-\[Alpha], -\[Beta]}};
    V1 = Table[
            Tau^Mod[((PowerMod[F1[[1, 2]], -1, db])*(F1[[1, 1]]*(s^2) -
                    2*r*s + F1[[2, 2]]*(r^2))), db] ket[r].bra[s], {r, 0,
                d - 1}, {s, 0, d - 1}]/Sqrt[d];
    V2 = Table[
            Tau^Mod[((PowerMod[F2[[1, 2]], -1, db])*(F2[[1, 1]]*(s^2) -
                    2*r*s + F2[[2, 2]]*(r^2))), db] ket[r].bra[s], {r, 0,
                d - 1}, {s, 0, d - 1}]/Sqrt[d];
    VF1 = Total[Total[V1]];
    VF2 = Total[Total[V2]];
    UF = VF1.VF2;
        )]
UF // MatrixForm
```


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[^0]:    3.1 Representation of the SIC vectors on the Bloch Sphere forming the vertices of dual tetrahedra inscribed within a cube27

[^1]:    ${ }^{1}$ Pronounced seek.

