Negative specific heat in self-gravitating systems

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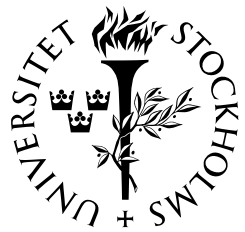

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To Maria, Jakob and Erika

Abstract

This essay is focused on identifying conditions for the presence of negative specific heat in some special models of self-gravitating systems and similar systems of attracting particles.

An introduction to elementary non-relativistic gravitation and to statistical physics is given. A distinction between long- and short-range forces is defined. The Virial theorem is explained. An introduction to statistical physics applied to non-relativistic gravitation is given. Calculations on a couple of special models of self-gravitating systems are presented, especially of the specific heat in those systems. Conclusions are drawn about the cause of negative specific heat in self-gravitating systems. This is done in a chapter that mainly presents an article written on the subject. This chapter contains the most relevant information to understand the conclusions, and can be read without reading the other chapters before.

The tool used in this essay to investigate the cause of negative specific heat in self-gravitating systems, is to analyse four different models of systems of attracting particles, and to map the sign of the specific heat for different combinations of the number of spatial dimensions of the system, $D(\geq 3)$, and the exponent, $\nu(\neq 0)$, in the force potential, $\phi = Cr^{\nu}$. Negative specific heat in such systems is found to be present exactly for $\nu = -1$. For many combinations of D and ν representing long-range forces, the specific heat is positive or zero, for all four models. The impression given by the literature is, however, that negative specific heat in self-gravitating systems arises because of the long-range nature of the gravitational force. This description is here challenged.

Preface

This essay was written as a Master Thesis during 2003 and 2004 at the Department of Physics at Stockholm University. During this period, I got insight into the fascinating area of statistical physics, and its application to the somewhat peculiar self-gravitating systems. I felt that in this area different subjects of great interest meet: Conceptual questions about statistical physics, the powerful Virial theorem, exciting properties of systems such as nonextensivity, negative specific heat and phase-transitions, descriptions in higher dimensions (in this essay), short- and long-range forces, and the formation of astrophysical objects.

In an essay like this one, there preferably should be both a part that refers to existing knowledge of the area, and a part where research done by the author is presented. This essay is meant both as an introduction to the area of statistical physics applied to self-gravitating systems, and as a presentation of the research results achieved during this thesis work. In the chapter "Introduction", the different parts of the essay are overviewed. The work with the essay was at first governed by learning the methods of the area, mainly by reading articles. During this period, also some ideas occured about what to make research about. Some months later, this led to the interesting results described in the "Conclusions" chapter.

I want to thank my supervisor, professor Ingemar Bengtsson, for kind and experienced supervision, for listening to my frequent presentations of what I've recently learned, for telling me about useful articles in the area, and for constructive feedback about what's a good idea and what's not. I also want to thank those who helped me with reading the texts for correction of errors. I especially want to thank Maria for patience with me sitting working at any time.

In the end, I feel very comfortable with having done this work. It has given me knowledge, I got the chance to do some research — and I had a good time!

Stockholm, April 5, 2004

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Contents

Chapter 1

Introduction

A self-gravitating system is a system that is defined by the gravitational interaction between its parts. This essay is an introduction to the area of statistical physics of self-gravitating systems in internal equilibrium, and a presentation of the research made in that area by the author.

The basic equations of non-relativistic gravitation as well as introductory statistical physics is presented for those who want to refresh their knowledge, as well as to learn about gravitation in different number of dimensions (≥ 2). This is covered within chapter 2 and chapter 3. For a broader course in statistical physics, see Goodstein [6]. There are a lot of conceptual difficulties related to the foundations of statistical physics. Some of them are about the irreversibility of time. This subject is covered, from different points of view, in Davies [5].

The physics of gravitating systems is presented in Saslaw [16] and Binney/Tremaine [3]. When trying to apply statistical physics to gravitating systems, one encounters difficulties related to the non-extensivity of these systems. An overview of different approaches to this matter is presented in Padmanabhan [15], and in chapter 6. Non-extensivity occurs because of the long-range nature of the gravitational force. A definition of long- and short-range forces is given in chapter 4.

A powerful tool for analysing different kind of systems is the Virial theorem. An advanced course in the application of this theorem to astrophysical systems is given in Collins [4]. A pedagogical introduction to the theorem is given in chapter 5.

To overcome difficulties with analysing realistic self-gravitating systems, some "toy models" of such systems have been invented, models where some of the properties of the realistic system have been changed, to simplify mathematical analyses. One fundamental such model is the "isothermal sphere". The stability and instability of this system has first been analysed by Antonov [1], and in other ways by several others, see Meylan/Heggie [13, page 93-96]. This model is covered in chapter 6.8 and chapter 7.1. A model that contains many particles but whose geometry makes it easy to analyse, was constructed by Lynden-Bell/Lynden-Bell [11], and is analysed also in higher dimensions in chapter 7.5. A variation on the isothermal sphere, with a dramatic phase-transition, was constructed by Aronson/Hansen [2], and is covered in chapter 7.7. A model with a non-gravitational potential, was constructed by Thirring [17], to show that also other non-extensive systems than gravitating ones possess negative

specific heat. The model is covered in chapter 7.6. Hertel/Thirring [7] extends the analysis of this kind of model. The "binary star model" presented in Padmanabhan [15], and also in higher dimensions in chapter 7.4, contains only two particles, and is easy to analyse. In chapter 7.2 the "Virial model", and in chapter 7.3 the "Circular orbit model", in arbitrary dimensions, are presented.

Negative specific heat in self-gravitating systems has been investigated, among others, by Hertel/Thirring [7], Hut [8], Lynden-Bell [10], Lynden-Bell/Lynden-Bell [11], Lynden-Bell/Wood [12], Padmanabhan [15] and Thirring [17].

Since this essay is focused on the specific heat of self-gravitating systems, where there is a temperature defined for the complete system, only systems in internal equilibrium are considered in this essay. The very extensive area of relaxation, describing the evolution of a system that is not yet in internal equilibrium, is not covered in this essay. For a short glimpse into relaxation, see chapter 6.2.

The scope of the research presented in this essay, is to analyse under what circumstances systems of attracting particles possess negative specific heat. The tool to do that here, is to analyse some of the special models of such systems in arbitrary dimensions (≥ 3), and with arbitrary exponent ($\neq 0$) in the force potential. The interesting conclusions of this analysis is presented in chapter 8.

All analyses in higher dimensions in this essay, have been made by the author. The Circular orbit model has been invented by the author, but may have been presented by others in sources not known by the author.

To let the interested reader explore related areas of statistical physics and gravitation, we here present some additional sources. Oppenheim [14] makes an attempt to create a theoretical frame and notation to handle non-extensive systems, especially in general relativity. The related area of black hole thermodynamics is reviewed in Wald [19]. Tsallis [18] considers other, more general, ways of defining entropy, that may be useful when studying certain kinds of systems.

Chapter 2

Non-relativistic gravitation

The potential energy of a particle with mass m in a gravitational potential ϕ is

$$E_p(\vec{x},t) = m\phi(\vec{x},t) \tag{2.1}$$

where $\vec{x}$ is the position vector for the particle and t the time parameter. The potential ϕ is also called a "gravitational field". The force on the particle due to that field is

$$\vec{F}(\vec{x},t) = -m\vec{\nabla}\phi(\vec{x},t) \tag{2.2}$$

The most general equation describing a non-relativistic gravitational field ϕ is a differential equation called "Poisson's equation". It relates the gravitational field to the mass density, ρ, by

$$\vec{\nabla}^2\phi = 4\pi G\rho \tag{2.3}$$

where G is the gravitational constant. ρ is a function of the position coordinates and possibly also of the time parameter. Since the "$\vec{\nabla}$" operator only depends on the space parameters, there will, for each point in time, exist a solution, ϕ, to (2.3) that emerges only from the space dependence of ρ at that particular time. The time dependence of ϕ will emerge from the time dependence inherent in ρ.

Of particular interest is the solution to (2.3) when

$$\rho = M\delta(\vec{x} - \vec{x}_0) \tag{2.4}$$

where δ is the "Dirac delta-function". This represents the situation when the field $\phi(\vec{x})$ emerges from a point mass M at the location $\vec{x}_0$. We will now investigate the form of ϕ for this particular case. The properties of the delta-function give two demands on ϕ. At first

$$\vec{\nabla}^2\phi(\vec{x}) = 0 \qquad \vec{x} \neq \vec{x}_0 \tag{2.5}$$

This is valid for

$$\phi(\vec{x}) = \frac{C}{|\vec{x} - \vec{x}_0|} \qquad \vec{x} \neq \vec{x}_0 \tag{2.6}$$

as easily can be seen by evaluating the left-hand side of (2.5):

$$\begin{aligned}\vec{\nabla}^2\phi &= C\sum_{i=1}^{3}\frac{\partial^2}{\partial x_i^2}(\sum_{j=1}^{3}(x_j-x_{j0})^2)^{-\frac{1}{2}} = \\ &= C\sum_{i=1}^{3}(-(\sum_{j=1}^{3}(x_j-x_{j0})^2)^{-\frac{1}{2}} + 3(x_i-x_{i0})^2(\sum_{j=1}^{3}(x_j-x_{j0})^2)^{-\frac{3}{2}}) = \\ &= 0 \end{aligned} \tag{2.7}$$

In spherical coordinates, with origo at $\vec{x}_0$, the same calculation, (2.7), can be expressed in an even easier way, with

$$\phi(r) = \frac{C}{r} \qquad\qquad r \neq 0 \tag{2.8}$$

and, since ϕ does not depend on the angular coordinates

$$\vec{\nabla}^2\phi = \vec{\nabla}_r^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}\phi) = \frac{1}{r^2}\frac{\partial}{\partial r}(-C) = 0 \qquad\qquad r \neq 0 \tag{2.9}$$

The second criterion that ϕ has to fulfil comes from (2.4) and integrating the equation (2.3) over a volume of space containing $\vec{x}_0$.

$$\begin{aligned} 4\pi GM &= \int \mathrm{d}V_{\vec{x}_0}\, 4\pi GM\delta(\vec{x}-\vec{x}_0) = \\ &= \int \mathrm{d}V_{\vec{x}_0}\, \vec{\nabla}^2\phi = \int \mathrm{d}V_{\vec{x}_0}\, \vec{\nabla}\cdot\vec{\nabla}\phi = \oint \mathrm{d}\vec{S}\,\vec{\nabla}\phi \end{aligned} \tag{2.10}$$

where the last step comes from Gauss' theorem. $\mathrm{d}\vec{S}$ is an infinitesimal surface element on the closed surface S, surrounding the point $\vec{x}_0$. The value of the integral is independent of the choice of S, as long as S is closed and surrounds $\vec{x}_0$. Let us choose S as a sphere with centre at $\vec{x}_0$ and with radius r. We also use (2.6) that gives

$$\vec{\nabla}\phi = -\frac{C}{|\vec{x}-\vec{x}_0|^2} = -\frac{C}{r^2} \qquad\qquad \vec{x} \neq \vec{x}_0 \qquad r \neq 0 \tag{2.11}$$

We can then write (2.10) as

$$4\pi GM = 4\pi r^2(-\frac{C}{r^2}) \tag{2.12}$$

(2.12) is satisfied if $C = -GM$. We obtain the result that a point mass M at $\vec{x}_0$ results in a gravitational field

$$\phi(\vec{x}) = -\frac{GM}{|\vec{x}-\vec{x}_0|} \qquad\qquad \vec{x} \neq \vec{x}_0 \tag{2.13}$$

It is relevant to remark that (2.13) is the only possible solution for Poisson's equation (2.3) in the point mass case. There is no other function than (2.13) that fulfils both (2.7) and (2.10). To describe this kind of matter in an elegant way applicable to any linear differential equation, it is convenient to use the concept of "Green's functions". Let $L(\vec{x})$ be a linear differential operator, and

$$L(\vec{x})u(\vec{x}) = f(\vec{x}) \tag{2.14}$$

Let us assume that there exists an inverse to $L(\vec{x})$, and that it has the following form

$$u(\vec{x}) = L^{-1}(\vec{x})f(\vec{x}) = \int \mathrm{d}V_{\vec{x}'}\, \mathcal{G}(\vec{x},\vec{x}')f(\vec{x}') \tag{2.15}$$

It is possible to show that this inverse exists. The operator $\mathcal{G}(\vec{x},\vec{x}')$ works in analogy to a matrix that transforms a vector (here: a function) to another vector. $\mathcal{G}$ is called a Green's function. We have

$$\begin{aligned} f(\vec{x}) &= L(\vec{x})L^{-1}(\vec{x})f(\vec{x}) = \\ &= L(\vec{x}) \int \mathrm{d}V_{\vec{x}'}\, \mathcal{G}(\vec{x},\vec{x}')f(\vec{x}') = \\ &= \int \mathrm{d}V_{\vec{x}'}\, (L(\vec{x})\mathcal{G}(\vec{x},\vec{x}'))f(\vec{x}') \end{aligned} \tag{2.16}$$

This identity holds for all f if

$$L(\vec{x})\mathcal{G}(\vec{x},\vec{x}') = \delta(\vec{x}-\vec{x}') \tag{2.17}$$

In our case

$$\mathcal{G}(\vec{x},\vec{x}') = -\frac{GM}{|\vec{x}-\vec{x}'|} \tag{2.18}$$

An important property of (2.3) is its linearity. That means, that if

$$\vec{\nabla}^2\phi_i = 4\pi G\rho_i \qquad\qquad i = 1,2,\ldots,N \tag{2.19}$$

then

$$\vec{\nabla}^2 \sum_{i=1}^{N} C_i\phi_i = 4\pi G \sum_{i=1}^{N} C_i\rho_i \tag{2.20}$$

where the C_i:s are constants.

If we combine (2.2) and (2.8), we get

$$\vec{F} = -\frac{GmM}{|\vec{x}-\vec{x}_0|^2}\hat{r} \tag{2.21}$$

which is known as "Newton's law of gravitation".

In this essay, we will be interested in models of self-gravitating systems in different number of dimensions. Therefore, we have to look at the implications of this when considering the form of the solutions to Poisson's equation (2.3). In three dimensions we got (2.13). In D dimensions it is correct to use the following expression for $\vec{\nabla}$:

$$\vec{\nabla}_D = \sum_{i=1}^{D} \hat{x}_i \frac{\partial}{\partial x_i} \tag{2.22}$$

(2.3) becomes, in D dimensions

$$\vec{\nabla}_D^2\phi(\vec{x}_D) = 4\pi G\rho(\vec{x}_D) \tag{2.23}$$

where

$$\vec{x}_D = \sum_{i=1}^{D} \hat{x}_i x_i \tag{2.24}$$

We are interested in solutions, ϕ, in the case of a point charge at $\vec{x}_{D0}$. For $\vec{x}_D \neq \vec{x}_{D0}$ this is in analogy to (2.7). We suppose that the solution is of the form

$$\begin{aligned} \phi(\vec{x}_D) &= C_D |\vec{x}_D - \vec{x}_{D0}|^\nu = C_D r^\nu \\ r \neq 0 \qquad \nu \neq 0 \end{aligned} \tag{2.25}$$

For $D = 3$ and $\nu = -1$ we have the three-dimensional case. It is possible to show that the radial component of the nabla operator in D dimensions is

$$\vec{\nabla}^2_{Dr} = \frac{1}{r^{D-1}} \frac{\partial}{\partial r} r^{D-1} \frac{\partial}{\partial r} \tag{2.26}$$

In D dimensions we have

$$\begin{aligned} 0 = \vec{\nabla}^2_D \phi(r) = \vec{\nabla}^2_{Dr} C_D r^\nu = \\ = C_D \frac{1}{r^{D-1}} \frac{\partial}{\partial r} (r^{D-1} \frac{\partial}{\partial r} r^\nu) = C_D \nu \frac{1}{r^{D-1}} \frac{\partial}{\partial r} r^{D+\nu-2} = \\ = C_D \nu (D + \nu - 2) r^{\nu-2} \qquad r \neq 0 \qquad \nu \neq 0 \end{aligned} \tag{2.27}$$

In order for this to be true for all $r \neq 0$, $\nu \neq 0$ there is

$$\nu = 2 - D \tag{2.28}$$

But this also implies $D \neq 2$. For $D = 2$, we use

$$\phi(\vec{x}_2) = C_2 \ln |\vec{x}_2 - \vec{x}_{20}| \tag{2.29}$$

This potential satisfies Poisson's equation, (2.23), in the two-dimensional case, for $\vec{x}_2 \neq \vec{x}_{20}$. To determine the constant C_D in (2.25) and C_2 in (2.29), we calculate, in analogy with (2.10), with use of (2.23)

$$\begin{aligned} 4\pi GM = \int \mathrm{d}V_{\vec{x}_{D0}} \, 4\pi GM \delta(\vec{x}_D - \vec{x}_{D0}) = \int \mathrm{d}V_{\vec{x}_{D0}} \, \vec{\nabla}^2 \phi = \\ = \int \mathrm{d}V_{\vec{x}_{D0}} \, \vec{\nabla}_D \cdot \vec{\nabla}_D \phi = \oint \mathrm{d}\vec{S}_D \vec{\nabla}_D \phi \end{aligned} \tag{2.30}$$

where the integration is performed over a closed surface surrounding the point $\vec{x}_{D0}$. Let S_D be the area of a hypersphere with unit radius in a D-dimensional space. From (2.30), (2.25) and (2.28), we have for $D \geq 3$

$$\begin{aligned} 4\pi GM = \oint \mathrm{d}\vec{S}_D \, C_D \nu r^{\nu-1} \hat{r} = C_D \nu S_D r^{D-1} r^{\nu-1} = \\ C_D (2 - D) S_D \qquad D \geq 3 \end{aligned} \tag{2.31}$$

From (2.30) and (2.29), we have for $D = 2$

$$4\pi GM = \oint \mathrm{d}\vec{S}_D \, C_2 r^{-1} \hat{r} = C_2 2\pi r r^{-1} = C_2 2\pi \tag{2.32}$$

(2.31) and (2.32) give

$$C_D = \begin{cases} \frac{4\pi GM}{(2-D)S_D} & D \geq 3 \\ 2GM & D = 2 \end{cases} \tag{2.33}$$

Observe that a C_D that varies with D, is based on a convention that Poisson's equation always has the same constant, independent of D. The opposite, a constant C_D and a varying constant in Poisson's equation, would also be quite possible as a convention. For S_D, we have

$$S_D = \frac{2\pi^{\frac{\nu}{2}}}{\Gamma(\frac{\nu}{2})} \tag{2.34}$$

or, in another form

$$S_D = \begin{cases} \frac{2^{\frac{D+1}{2}}\pi^{\frac{D-1}{2}}}{(D-2)!!} & \text{D odd} \\ \frac{2\pi^{\frac{D}{2}}}{(\frac{D}{2}-1)!} & \text{D even} \end{cases} \tag{2.35}$$

Chapter 3

Statistical physics

The evolution of a classical system with N particles in a D-dimensional space is in principle determined by the laws of mechanics together with initial conditions for $2DN$ phase-space coordinates. But, we have no possibility to know the initial conditions for all particles in, for instance, a gas. To calculate the position of all particles, equations for $2DN$ coordinates have to be solved, which for systems with many particles is practically impossible. And small disturbances, or errors in the premises, will make the results less accurate as the time evolves. Both theoretical and experimental experience has shown that it is possible to use statistical methods to investigate the properties of such systems. Even though there are still some very important questions regarding the theoretical foundation of statistical physics, for instance the assumption of ergodicity, the methods work in a satisfactory manner for many systems.

For a course in statistical physics, for instance look at Goodstein [6].

In statistical physics there is an important distinction between "macro-states" and "micro-states". Different macro-states are states distinguishable by macroscopic measurements, such as measurements of energy, temperature or the number of particles of a system. A micro-state is defined by the combination of the states of all particles in the system, and usually there is no possibility to determine this kind of state by measurement. But it is often possible by theory to relate any micro-state to the constraints on the macro-state, for instance a certain energy. There are usually many micro-states consistent with one macro-state, and especially with some of the macro-states. Therefore statistical treatment of the micro-states often result in deterministic macroscopic descriptions.

It is common to organize the theory in accordance to the constraints of the system. The simplest set of constraints is that the system has constant energy, E, constant number of particles, N, and occupies a constant volume in space, V. This kind of system is said to be part of a "micro-canonical ensemble". An ensemble is an imagined collection of a very huge number of systems that do not interact with each other. The ensemble then acts as a huge statistical material from which conclusions about single systems can be made. If we change the condition of constant energy to a condition of constant temperature, T, we get a system in the "canonical ensemble", with parameters T, N and V. If we also change the condition of constant number of particles, to a condition of constant chemical potential, μ, we get a system in the "grand canonical ensemble". We

will discuss the micro-canonical and the canonical ensembles, and investigate some parameters of interest. Observe that the canonical ensemble can not be applied to systems with negative specific heat without some modifications, see chapter 6.4.

One of the most important concepts in statistical physics and thermodynamics is "entropy". That is, because it is one of the principles of thermodynamics that the entropy of a completely isolated system (a system in the micro-canonical ensemble) increases, until it reaches its maximum value possible under the constraints on the system. The constraints may be a certain energy, number of particles or volume. Even when considering systems that are not in the micro-canonical ensemble, the properties of entropy are crucial to build the theory. Entropy is a macroscopic parameter, and was originally in terms of thermodynamics discovered by Clausius, and later formulated in statistical terms by Boltzmann (and statistically for black-body radiation by Planck). Boltzmann's definition is

$$S = k \ln \Omega \tag{3.1}$$

where k is Boltzmann's constant, and Ω the number of micro-states corresponding to the system's macro-state. The micro-states in this definition are presumed to occur with equal probability. This is valid for isolated systems. It is, however, from Boltzmann's formula, possible to derive a formula for the entropy where each micro-state is assigned a specific probability to occur. This formula is more generally applicable, and is regarded as the general definition of entropy. It reads

$$S = -k \sum_i p_i \ln p_i \tag{3.2}$$

where p_i is the probability of micro-state i, provided all micro-states counted for are compatible with the constraints on the macro-state. To derive the formula, we use the concept "ensemble". Suppose we have n number of identical systems, with the same macroscopic properties, but whose micro-states are independent of each other, and let n be huge. If system j's micro-state, i_j, occurs with probability p_i, then, since n is huge

$$n_i = np_i \tag{3.3}$$

where n_i is the number of systems in the ensemble that is in micro-state i. We let the macroscopic state of the ensemble be defined by the ordered set of these n_i:s. Let us also look at the ordered sequence of micro-state numbers, i_j, for system one, system two, and so on. A micro-state of the ensemble can be defined by a unique sequence of these numbers. The number of different micro-states possible for a certain macro-state of the ensemble, Ω_{ens}, can be calculated by considering the number of ways to order these numbers in a distinguishable way, with conserved n_i:s.

$$\Omega_{ens} = \frac{n!}{n_1! n_2! \ldots n_n!} \tag{3.4}$$

Since all the micro-states of the ensemble occur with equal probability once the probabilities p_i are given, we may use Boltzmann's entropy definition, (3.1), on the ensemble.

$$S_{ens} = k \ln \Omega_{ens} \tag{3.5}$$

Using (3.5) with Stirling's formula

$$\ln x! \approx x \ln x \tag{3.6}$$

which is valid for huge x, the formula for total probability, $\sum p_i = 1$, (3.4) and (3.3), we get

$$\begin{aligned} S_{ens} &= k(\ln n! - \sum_i \ln n_i!) = k(n \ln n - \sum_i n_i \ln n_i) = \\ &= k(n \ln n - \sum_i n p_i \ln n p_i) = k(n \ln n - n(\sum_i p_i) \ln n - n \sum_i p_i \ln p_i) = \\ &= -kn \sum_i p_i \ln p_i \end{aligned} \tag{3.7}$$

The systems in the ensemble are by definition independent of each other, which makes it possible to relate Ω_{ens} in a simple manner to the number of equal probability micro-states of one of the systems, Ω. This is an imagined parameter, since the system does not necessarily have such states. One possible way to relate the p_i:s to Ω is by

$$p_i = \frac{\lambda_i}{\Omega} \tag{3.8}$$

where $\sum \lambda_i = \Omega$. That means, we allow different imagined equal probability micro-states to represent the same true micro-state of the system. The relation between Ω_{ens} and Ω is

$$\Omega_{ens} = \Omega^n \tag{3.9}$$

which, according to (3.5), gives for the ensemble's entropy, S_{ens}, in relation to a system's entropy, S

$$S_{ens} = nS \tag{3.10}$$

(3.10) together with (3.7) give our general expression for the entropy, (3.2). In the case of equal probabilities, we set

$$p_i = \frac{1}{\Omega} \tag{3.11}$$

and we get Boltzmann's definition of entropy, (3.1).

In the micro-canonical ensemble, the energy is one of the constants of the system. The theory of this ensemble is applicable, even though this constant may vary, but in such a slow way that the system is in internal equilibrium all the time. In the canonical ensemble, the temperature is one of the constants of the system. Also here it may vary, slowly enough to keep the equilibrium. To study the properties of systems in this ensemble, we use the concept of a total system at temperature T, isolated from its surrounding. The system has constant energy E and has entropy S. A system under observation called "system 1" is part of the total system, and has energy E_1 and entropy S_1. Let the part of the total system that is not included in system 1, be called "system 2", and let it have energy E_2 and entropy S_2. The only interaction between system 1 and system 2 is supposed to be the exchange of thermal energy. Then there is

$$E = E_1 + E_2 \tag{3.12}$$

$$S = S_1 + S_2 \tag{3.13}$$

We are interested in E_1 and S_1, since these parameters characterize system 1. Since the total system is isolated and in equilibrium it will be in a state that represents a maximum in entropy when varying internal parameters of the system. This implies that E_1 and S_1 will take values that maximize this entropy. We have from (3.12) and (3.13) and the statement about maximum entropy

$$0 = \frac{\partial S}{\partial E_1}|_{E,V,N,V_1,N_1} = \\ = \frac{\partial S_1}{\partial E_1}|_{V_1,N_1} + \frac{\partial S_2}{\partial E_2}|_{V_2,N_2}\frac{\mathrm{d}E_2}{\mathrm{d}E_1} = \frac{\partial S_1}{\partial E_1}|_{V_1,N_1} - \frac{\partial S_2}{\partial E_2}|_{V_2,N_2} \qquad (3.14)$$

where the entities at the vertical bar are held constant under differentiation. We see that the quantity $\frac{\partial S_i}{\partial E_i}|_{V_i,N_i}$ is equal between systems that is in thermal equilibrium to each other. This may remind us about the parameter "temperature", that also has this property. As a matter of fact, it is a convention to define the temperature of a system as

$$T = (\frac{\partial S}{\partial E}|_{V,N})^{-1} \qquad (3.15)$$

The exact form of (3.15) is consistent with a formula for the entropy of a perfect gas, derived from thermodynamic relations, and where the temperature is a parameter. Observe that also systems in the micro-canonical ensemble have temperature, since (3.15) can be applied on these systems too. Now, let the system 1 be much smaller (in energy) than system 2. In relation to system 1, system 2 is called a "heat bath". This concept is often used to analyse some properties of the canonical ensemble, because in this situation fluctuations in thermal energy of system 1 do not significantly change the temperature of system 2. The total system is isolated. We also, approximately, regard system 2 as isolated, since the exchange of heat with system 1 (due to fluctuations) is small compared to the total heat energy of system 2. We can then assume that system 2 has a number of equal probability micro-states, Ω_2, corresponding to its energy, E_2. We can then use (3.1) as the expression for the entropy. We also use (3.12) for the energy, the definition of temperature (3.15), and look at the situation when system 1 is in micro-state i, with an energy denoted E_i.

$$k\ln\Omega_2(E_2) = S_2(E_2) = S_2(E - E_i) = \\ = S_2(E) - E_i\frac{\partial S_2}{\partial E}|_{V,N} + \frac{E_i^2}{2}\frac{\partial^2 S_2(E)}{\partial E^2}|_{V,N} + \ldots = \\ = S_2(E) - \frac{E_i}{T} + \frac{E_i^2}{2}\frac{\partial}{\partial E}\frac{1}{T} + \ldots \approx S_2(E) - \frac{E_i}{T} \qquad (3.16)$$

The last step is valid, because system 2 does not change temperature significantly when exchanging energy of the order of E_i, according to the definition of "heat bath". We have then, for the probability of micro-state i:

$$p_i \propto \Omega_2(E_0 - E_i) = \exp(\frac{1}{k}k\ln\Omega_2) \approx \\ \approx \exp(\frac{S_2(E)}{k})\exp(-\frac{E_i}{kT}) = \exp(\frac{S_2(E)}{k})\exp(-\beta E_i) \qquad (3.17)$$

where we have used the definition

$$\beta \equiv \frac{1}{kT} \qquad (3.18)$$

We normalize the probability in (3.17).

$$p_i = \frac{\exp(\frac{S_2(E)}{k})\exp(-\beta E_i)}{\sum_i \exp(\frac{S_2(E)}{k})\exp(-\beta E_i)} = \frac{\exp(-\beta E_i)}{\sum_i \exp(-\beta E_i)} = \frac{\exp(-\beta E_i)}{Z} \tag{3.19}$$

The formula for p_i is called a "Boltzmann probability distribution". In (3.19), we used the definition

$$Z \equiv \sum_i \exp(-\beta E_i) \tag{3.20}$$

The function Z is called the "partition function" of the system. In (3.19) and (3.20), we have summed over every micro-state. We may state a corresponding formula, but where we sum over all energies, once per energy level. If we let $g(E_i)$ denote the number of micro-states with energy E_i, and p_{E_i} the probability of the energy level E_i, we get

$$p_{E_i} = \frac{g(E_i)\exp(-\beta E_i)}{Z} \tag{3.21}$$

where

$$Z = \sum_{E_i} g(E_i)\exp(-\beta E_i) \tag{3.22}$$

For a system in the canonical ensemble, the energy will fluctuate, even though the temperature is constant. The mean energy can be calculated by use of the partition function, Z.

$$< E >= -\frac{\partial \ln Z}{\partial \beta}|_{V,N} \tag{3.23}$$

where "<>" expresses a time average. It is easy to see that this derivative results in a weighted average of the energies of all micro-states. For a system in the micro-canonical ensemble, i.e. an isolated system, (3.1) gives the entropy. For a system in the canonical ensemble, i.e. a system in contact with a heat bath, we use the general definition of entropy, (3.2), together with the Boltzmann probability distribution, (3.19).

$$\begin{aligned} S(T,V,N) &= -k\sum_i p_i \ln p_i = -k\sum_i (\frac{\exp(-\beta E_i)}{Z}\ln\frac{\exp(-\beta E_i)}{Z}) = \\ &= -k(\frac{\sum_i(-\beta E_i)\exp(-\beta E_i)}{Z} - \frac{\ln Z\sum_i \exp(-\beta E_i)}{Z}) = \\ &= \frac{< E >}{T} + k\ln Z \end{aligned} \tag{3.24}$$

In the micro-canonical ensemble, equilibrium implied a maximum in the entropy. It can, with some effort, be derived that in the canonical ensemble, equilibrium implies a minimum in a quantity called the "Helmholtz free energy". It is defined as

$$F \equiv -kT\ln Z \tag{3.25}$$

By combining (3.24) and (3.25), we get the relation

$$F =< E > -TS \tag{3.26}$$

We will now look at a general formula that can be applied to the kind of systems we have been discussing. As mentioned before, the grand canonical

ensemble involves fluctuating energy and fluctuating number of particles. This ensemble is more general than the canonical ensemble where only the energy may fluctuate. Therefore, we will here give a formula for the grand canonical ensemble that relates the parameters of this ensemble to each other.

$$\mathrm{d}E = T\mathrm{d}S - P\mathrm{d}V + \mu\mathrm{d}N \tag{3.27}$$

μ is the chemical potential and N the number of particles. Here, we have also included a term with the pressure, P, and the volume, V. This term can also be present in the canonical ensemble, but in this essay we are mostly interested in systems with constant volume, where this term vanish. The formula (3.27) is obtained by regarding S to be a function of E, V and N, and by differentiation of S in regard to these parameters. Also the definitions of temperature, pressure and chemical potential are used.

When studying the thermodynamical properties of systems, it is often useful to look at the relation between energy, E, and temperature, T. This relation can say some things about the behaviour of these systems. The most natural parameter to study in this sense is the "specific heat" of the system. The specific heat is defined as

$$C_V \equiv \frac{\partial < E >}{\partial T}|_V = -k\beta^2 \frac{\partial < E >}{\partial \beta}|_V \tag{3.28}$$

This is for constant volume. There is also a specific heat for constant pressure, C_P, but we will not be interested in this, in this essay. When considering systems with constant particle number, (3.28) can be written

$$C_V = \frac{\partial < E >}{\partial T}|_{V,N} = -k\beta^2 \frac{\partial < E >}{\partial \beta}|_{V,N} \tag{3.29}$$

and we have

$$C_V = \frac{\partial < E >}{\partial T}|_{V,N} = \frac{\partial < E >}{\partial S}|_{V,N} \frac{\partial S}{\partial T}|_{V,N} = kT \frac{\partial S}{\partial T}|_{V,N} \tag{3.30}$$

where we have used the definition of temperature, (3.15). For systems where S is a function of $< E >$, it is easier to define

$$f(< E >) = \frac{1}{\frac{\partial S}{\partial < E >}|_{V,N}} = T \tag{3.31}$$

where we also have used the definition of temperature, (3.15). We then have, by combining (3.28) and (3.31)

$$C_V = (\frac{\partial f(< E >)}{\partial < E >}|_V)^{-1} \tag{3.32}$$

It is possible to calculate the specific heat, C_V, from the partition function, Z, of a system in the canonical ensemble. Earlier, in (3.23), we expressed the mean energy, $< E >$, in terms of the partition function, Z. Together with the expression for the specific heat restricted to systems with constant particle number, (3.29), we get for C_V

$$C_V = k\beta^2 \frac{\partial^2 \ln Z}{\partial \beta^2}|_{V,N} \tag{3.33}$$

From this expression, it is possible to derive a relation between the specific heat and the variance of the energy fluctuations of the system.

$$C_V = k\beta^2 \frac{\partial^2 \ln Z}{\partial \beta^2}|_{V,N} = k\beta^2 (\frac{\frac{\partial^2 \ln Z}{\partial \beta^2}|_{V,N}}{Z} - (\frac{\frac{\partial \ln Z}{\partial \beta}|_{V,N}}{Z})^2) =$$
$$= k\beta^2 (< E^2 > - < E >^2) = k\beta^2 < (E - < E >)^2 > \qquad (3.34)$$

Observe that (3.34) implies that the specific heat of a system in the canonical ensemble is positive.

Let us now investigate formulas to apply the theory to a system of N particles with energy E, in the micro-canonical ensemble. The system is assumed to have a phase-space of volume $g(E)$. To apply statistical physics to this kind of system, it is a standard assumption that $g(E)$ is proportional to Ω (the number of micro-states of the system). To know the constant of proportionality one uses an assumption that each quantum-state is one micro-state, and Planck's constant appears in the equations. It is however not necessary to know this constant of proportionality, and we will here omit it. Let us call the Hamiltonian of the system H. Then the phase-space volume can be calculated by

$$g(E) = \mathrm{d}E \int (\Pi_{i=1}^{n_s} \mathrm{d}x_{Di}) (\Pi_{i=1}^{n_m} \mathrm{d}p_{Di}) \, \delta(E - H(x_{Di}, p_{Di})) \qquad (3.35)$$

where we assume the system to have n_s space degrees of freedom and n_m momentum degrees of freedom. The delta-function ensures that only the parts of the phase-space that is consistent with the constraint on the energy of the system contributes to the volume. For those systems that we will investigate in this essay, the Hamiltonian can be written on the following form.

$$H = E_p + E_k = E_p(x_{Di}) + \sum_{i=1}^{n_m} \frac{p_{Di}^2}{2m_i} \qquad (3.36)$$

where E_p is the potential energy, E_k the kinetic energy, and m_i the mass of paricle i. (3.35) can, with use of (3.36), be written

$$g(E) = \mathrm{d}E \int (\Pi_{i=1}^{n_s} \mathrm{d}x_{Di}) (\Pi_{i=1}^{n_m} \mathrm{d}p_{Di}) \, \delta(E - E_p(x_{Di}) - \sum_{i=1}^{n_m} \frac{p_{Di}^2}{2m_i}) \qquad (3.37)$$

When integrating over the momentum parameters, we integrate over all combinations of the those parameters that satisfy that the sum in the argument of the delta-function equals $E - E_p$. These combinations constitutes a surface of a sphere (or to be correct: an ellipsoid, see below) with radius $(E - E_p)^{\frac{1}{2}}$ in a space with n_m dimensions. The area of this surface is $C_1 (E - E_p)^{\frac{n_m}{2} - 1}$, where C_1 is a constant depending only of n_m. We also have to correct the surface area with the m_i:s, because they scale the area (and deforms the sphere to be an ellipsoid). We can then write (3.37)

$$g(E) \propto \mathrm{d}E \int (\Pi_{i=1}^{n_s} \mathrm{d}x_{Di}) (\Pi_{j=1}^{n_m} m_j^{\frac{1}{2}}) \, (E - E_p(x_{Di}))^{\frac{n_m}{2} - 1} \qquad (3.38)$$

where we have dropped all unnecessary constants. We keep the m_i:s, because for one of our models we will study, mathematically a spatial coordinate can

be included in them. So they are not necessarily masses, but represent some function appearing in the denominator of the sum in the expression for the kinetic energy. Often, E_p is a function of one radial coordinate, and (3.38) can be written

$$g(E) \propto \mathrm{d}E \int (\Pi_{i=1}^{n_s} \mathrm{d}x_{Di}) \, (\Pi_{j=1}^{n_m} m_j^{\frac{1}{2}}) \, (E - E_p(r))^{\frac{n_m}{2}-1} \tag{3.39}$$

This radial coordinate does not necessarily be radial in relation to all the other space coordinates. Some of the other coordinates may be, for instance, Cartesian. Let us call the number of coordinates that are the angle coordinates associated to r, n_{sr}. Then (3.39) can be written

$$g(E) \propto \mathrm{d}E \int \mathrm{d}r \, r^{n_{sr}} (\Pi_{j=1}^{n_m} m_j^{\frac{1}{2}}) \, (E - E_p(r))^{\frac{n_m}{2}-1} \tag{3.40}$$

where we have omitted multiplicative constants, for instance the volume due to integration in the x_i degrees of freedom not being angle coordinates in relation to r. (3.40), we will use when we calculate on the special models in chapter 7.

It is possible to calculate the expectation value of any function of the phase-space coordinates by using (3.40) with the function in consideration within the integral. It is also possible to calculate the partition function, Z, for the case of a system in the canonical ensemble by using (3.40) in (3.22), if the sum in (3.22) is replaced by an integral.

We want to know how to calculate the specific heat at constant volume, C_V, for a system in the micro-canonical ensemble with phase-space volume $g(E)$. With use of (3.1), (3.31) and (3.32), we obtain

$$C_V = k \frac{g'(E)^2}{g'(E)^2 - g(E)g''(E)} \tag{3.41}$$

Chapter 4

Long- and short-range forces

We will here investigate a simple way to define a distinction between long- and short-range forces. In this essay we make most of our investigations on potentials of the form

$$\phi(\vec{x}_D) = C|\vec{x}_D - \vec{x}_{D0}|^\nu \tag{4.1}$$

In this general case it is possible to investigate, for any number of dimensions D, the consequence of forces with different ν. We will especially be interested in gravitation where, according to (2.28), ν depends on D like $\nu = 2 - D$. Also for other forces than gravitation, there may be relations between ν and D, but we do not look specifically on those forces. We just look at different combinations of ν and D. We will only consider situations with only one type of charge, like with gravitation or the van der Waals force. In other cases we will get shielding between the different kinds of charges, and the situation is different. The derivation is valid for non-gravitational forces if "mass" is interpreted as the relevant charge in consideration, ρ as the corresponding density, and the constant C as the constant applicable. The easiest way to distinguish between long-range and short-range forces is by looking at a continuous medium with constant density, and investigate from which areas the potential energy of a particle embedded in the media at the point $\vec{x}_D$ comes. If only regarding contributions from spherical shells of radius Δ (and with the particle in the centre), and where $\epsilon \leq \Delta \leq R$, the potential energy of the particle with mass m is

$$\begin{aligned} U(\vec{x}_D) &= \\ &= -\int_{V_\Delta} \mathrm{d}V_{D0}\, C'|\vec{x}_D - \vec{x}_{D0}|^\nu \rho(\vec{x}_{D0}) m \propto \int_\epsilon^R \mathrm{d}r_0\, r_0^{D-1} r_0^\nu = \\ &= \int_\epsilon^R \mathrm{d}r_0\, r_0^{D+\nu-1} \propto [r_0^{D+\nu}]_\epsilon^R = R^{D+\nu} - \epsilon^{D+\nu} \end{aligned} \tag{4.2}$$

where $D + \nu \neq 0$. We assume $\epsilon \ll 1$ and $R \gg 1$, giving a strong dependence of ϵ for negative $D + \nu$, and a strong dependence of R for positive $D + \nu$. With this method, short-range forces are characterized by

$$D + \nu < 0 \tag{4.3}$$

and long-range forces by

$$D + \nu > 0 \tag{4.4}$$

For gravitational forces we use (2.28), and we can conclude that gravitation, in any number of dimensions, is a long-range force according to the criteria under consideration.

Chapter 5

The Virial theorem

The "Virial theorem" relates time averages of some entities of a system to each other. In order to be valid, some premises have to be true, and we will discuss that later. In its most general form the theorem reads

$$< K >= -\frac{1}{2} < \sum_i \vec{F}_i \cdot \vec{r}_i > \tag{5.1}$$

where K is the kinetic energy of the system, $\vec{F}_i$ the force acting on particle i, and $\vec{r}_i$ the position of that particle. "<>" denotes a time average. The right-hand side of (5.1) is called "the Virial of Clausius". It can be applied to many different kinds of systems, and depending on the nature of the forces in the system of consideration, it results in a variety of formulas, describing different systems. See Collins [4] for a deep insight into the application of the Virial theorem to astrophysical systems. We will now deduce (5.1). To do that, we will look at the entity

$$B = \sum_i \vec{p}_i \cdot \vec{r}_i \tag{5.2}$$

where $\vec{p}_i$ is the linear momentum of particle i. We take the total time derivative of B.

$$\begin{aligned} \frac{\mathrm{d}B}{\mathrm{d}t} &= \sum_i \frac{\mathrm{d}\vec{r}_i}{\mathrm{d}t} \cdot \vec{p}_i + \sum_i \frac{\mathrm{d}\vec{p}_i}{\mathrm{d}t} \cdot \vec{r}_i = \\ &= \sum_i m_i \frac{\mathrm{d}\vec{r}_i}{\mathrm{d}t} \cdot \frac{\mathrm{d}\vec{r}_i}{\mathrm{d}t} + \sum_i \vec{F}_i \cdot \vec{r}_i = \\ &= \sum_i m_i v_i^2 + \sum_i \vec{F}_i \cdot \vec{r}_i = 2K + \sum_i \vec{F}_i \cdot \vec{r}_i \end{aligned} \tag{5.3}$$

where m_i is the mass, and v_i the velocity of particle i. We take the time average of this expression over a period τ.

$$\frac{1}{\tau}(B(\tau) - B(0)) = \frac{1}{\tau}\int_0^\tau \frac{\mathrm{d}B}{\mathrm{d}t}\mathrm{d}t \equiv < \frac{\mathrm{d}B}{\mathrm{d}t} >= 2 < K > + < \sum_i \vec{F}_i \cdot \vec{r}_i > \tag{5.4}$$

The derivation of the theorem is dependent on the possibility of setting $< \frac{\mathrm{d}B}{\mathrm{d}t} >$ to zero. There are two such possibilities. The first is that if the motion of

all particles are periodic, then if τ is taken to be the period of the complete system (if this period exists), the left-hand side of (5.4) will vanish. The second possibility is that if positions and velocities of all particles are finite, that means $|B|$ has an upper bound, then when making τ large enough, the left-hand side of (5.4) will be arbitrarily small. So, if we limit our considerations to systems which are periodic, or systems where the positions and velocities are finite, the Virial theorem, (5.1), is true. The theorem has many important applications, for instance in the kinetic theory of gases, or in stellar astrophysics. Since a stable system has finite positions and finite velocities, the Virial theorem can also be used to predict which system is stable, and which is not.

To make it possible to understand what the theorem implies, we take an example: a derivation of Boyle's law, also known as the equation of state for a perfect gas. First, we look at the left-hand side of (5.1). It is known from the equipartition theorem in statistical mechanics (see Goodstein [6]) that the time-averaged kinetic energy of a particle in a gas in a D-dimensional space is $\frac{D}{2}kT$, where k is Boltzmann's constant, and T the temperature. So, the time averaged kinetic energy of the system is

$$< K >= \frac{D}{2}NkT \tag{5.5}$$

where N is the number of particles in the system. Now, we look at the right-hand side of (5.1). The forces on the particles in the system may be forces between particles, for instance at collisions, and forces due to an externally applied potential. The forces on the particles in this system mainly comes from collisions with the walls. Let P be the pressure on the wall, $\mathrm{d}A$ a surface element of the wall, and $\hat{n}$ an outward-pointing normal to $\mathrm{d}A$. We then have

$$< \sum_i \vec{F}_i \cdot \vec{r}_i >= -P \int_A \mathrm{d}A\, \hat{n} \cdot \vec{r} \tag{5.6}$$

and by use of Gauss' theorem on (5.6), we obtain

$$< \sum_i \vec{F}_i \cdot \vec{r}_i >= -P \int_V \mathrm{d}V\, \vec{\nabla} \cdot \vec{r} = -DPV \tag{5.7}$$

where D is the number of spatial dimensions, and V the volume of the system. By combining (5.1), (5.5) and (5.7), we get

$$PV = NkT \tag{5.8}$$

We see that the equation of state for a perfect gas is independent of the number of dimensions. We may regard the case where we also have point-wise collisions between particles in the gas. At such a collision between some particles labelled i we have, according to the law of action and reaction

$$\sum_i \vec{F}_i \cdot \vec{r}_i = (\sum_i \vec{F}_i) \cdot \vec{r} = 0 \tag{5.9}$$

When calculating the Virial of Clausius for only this kind of force, we get

$$-\frac{1}{2} < \sum_i \vec{F}_i \cdot \vec{r}_i >= 0 \tag{5.10}$$

This means, since the Virial of Clausius is linear in $\vec{F}_i \cdot \vec{r}_i$, that a gas can be regarded as perfect even when the particles, except for collisions with the wall, also have point-wise interactions with each other.

Now, we will examine the application of the Virial theorem to a system of particles, interacting in pairs by a central force potential, and where particle i has mass m_i. We regard a potential due to particle j acting on particle i, on the form

$$V_{ij} = C m_i m_j r_{ij}^{\nu} \tag{5.11}$$

That is, V_{ij} is a homogeneous function in r_{ij}. This may be a gravitational potential in $D = 2 - \nu$ ($D \geq 3$) dimensions, see chapter 2. The force on particle i due to particle j is

$$\vec{F}_{ij} = C m_i m_j \nu r_{ij}^{\nu-2} (\vec{r}_j - \vec{r}_i) \tag{5.12}$$

where $\vec{r}_i$ and $\vec{r}_j$ is the position vector of particle i and j respectively. We see that

$$\vec{F}_{ij} = -\vec{F}_{ji} \tag{5.13}$$

Using (5.1), (5.12) and (5.13), we get for two particles with labels i and j respectively

$$\begin{aligned} < K >= & -\frac{1}{2}(\vec{F}_{ij} \cdot \vec{r}_i + \vec{F}_{ji} \cdot \vec{r}_j) = \\ & = \frac{1}{4}(\vec{F}_{ij} \cdot \vec{r}_j - \vec{F}_{ij} \cdot \vec{r}_i + \vec{F}_{ji} \cdot \vec{r}_i - \vec{F}_{ji} \cdot \vec{r}_j) = \\ & = \frac{1}{4}(\vec{F}_{ij} \cdot (\vec{r}_j - \vec{r}_i) + \vec{F}_{ji} \cdot (\vec{r}_i - \vec{r}_j)) = \frac{1}{4} \cdot 2\vec{F}_{ij} \cdot (\vec{r}_j - \vec{r}_i) = \\ & = \frac{1}{2} C m_i m_j \nu r_{ij}^{\nu} = \frac{\nu}{2} V \end{aligned} \tag{5.14}$$

where V is the potential energy of the two particles due to their interaction. Each pair of particles in the system contributes in the same way to the Virial of Clausius, and the potential energy due to their interaction will add to V. So, (5.14) is valid for the complete system, where V is the potential energy of the system. Observe that (5.14) is valid because the interaction between two particles has the form (5.11). The effective potential, arising from the system's density distribution, may be another one, but (5.14) is still valid. The total energy of the system is

$$E = V + K \tag{5.15}$$

(5.14) and (5.15) gives

$$< E >= \frac{\nu + 2}{\nu} < K > \tag{5.16}$$

and

$$< E >= \frac{\nu + 2}{2} < V > \tag{5.17}$$

Observe that an eventual dependence of the number of dimensions, D, only appears through ν, and only if there is a dependence between ν and D.

We will also investigate what happens if the system is restricted to the interior of a spherical box with centre in the central force potential's centre. This is a situation that appears when studying self-gravitating systems. The

contribution to the Virial of Clausius due to the force on the particles from this wall is, in analogy to (5.6) and (5.7)

$$-\frac{1}{2} < \sum_i \vec{F}_i \cdot \vec{r}_i >= \frac{1}{2} \int_A \mathrm{d}A\, P\hat{n} \cdot \vec{r} = \frac{1}{2} P \int_V \mathrm{d}V\, \vec{\nabla} \cdot \vec{r} = \frac{1}{2} DPV \tag{5.18}$$

Adding this contribution to (5.14) gives

$$< K >= \frac{\nu}{2} < V > + \frac{1}{2} DPV \tag{5.19}$$

Here, we have a direct dependence of D. (5.15) and (5.19) gives

$$< E >= \frac{\nu + 2}{\nu} < K > - \frac{D}{\nu} PV \tag{5.20}$$

and

$$< E >= \frac{\nu + 2}{2} < V > + \frac{1}{2} DPV \tag{5.21}$$

Chapter 6

Non-relativistic gravitation and statistical physics

A self-gravitating system is a system that is defined by its internal gravitational interaction between its parts.

6.1 Non-extensivity

A self-gravitating system exhibits some properties that make it difficult to use statistical methods from, for instance, the theory of gases. Gravitation is a long-range force, see chapter 4. This makes the energy of the system become a non-extensive parameter. That means, if we have two self- gravitating systems with energy E_1 and E_2 respectively, we have for the combined system in equilibrium

$$E \neq E_1 + E_2 \tag{6.1}$$

In a system with only short-range forces, the energy of the system due to interactions between subsystems depends on the area of the subsystems' surface, while the energy due to internal interactions in the subsystems scales as the volume of the subsystems. So, except for very small subsystems, the energy due to interactions between subsystems is negligible for short-range forces. In this case the energy is an extensive parameter. In the case of long-range forces the interaction energy scales as the volume of the subsystems. Then the interaction energy will remain important even for big subsystems. Hut [8] gives another view of the same phenomena. He takes an example with a self-gravitating system where the density (total mass per total volume) and temperature are held constant, and the mass (or number of particles), M, of the system is varied. Then $M \propto R^3$, where R is the radius of a limiting sphere. For the kinetic and potential energy we have $E_k \propto M$ and $E_p \propto \frac{M^2}{R} \propto M^{\frac{5}{3}}$. An extensive parameter growes linearly with the amount of substance in the system, in this case linearly with M. The total energy here has a component that grows faster than linearly, and is therefore a "superextensive" parameter. In an extensive system we have a "thermodynamic limit", that means, the value on the extensive observables divided by the amount of substance in the system is finite. Here, $\frac{E_p}{M}$ goes to infinity, and we have no thermodynamic limit.

An expression of the non-extensivity of these systems, is their inhomogeneity. That is, the density as well as the gravitational field varies over the system at internal equilibrium, and the energy of a subsystem depends on where in the system the subsystem resides.

6.2 Relaxation

To make a system come in internal thermodynamical equilibrium, there has to be some mechanism that relies on the gravitational interaction between objects. This mechanism is called "relaxation". There are two kinds of relaxations. Early in the life of star clusters, "violent relaxation" appears. This involves gravitational interaction between different groups of stars in the phase-space of the cluster. During this interaction, individual objects are scattered mainly by groups of objects, rather than by other individual objects. The groups are continuously reforming and objects change group. But the statistical distribution in phase-space lasts longer. In the end of this process, the system is no longer divided into groups. When the violent relaxation has faded, the "gentle relaxation" dominates. This process involves interaction between pairs of stars. At short distances the interaction is strong, resulting in large deflection and velocity change. But since there are more stars at large distances, the effect of the distant stars is bigger. The final state of gentle relaxation includes segregation of stars with different mass. The more massive objects accumulate at the center of the cluster. The time scale for the occurence of a nearly relaxed system is much shorter for violent relaxation than for gentle relaxation. For an insight to relaxation, for instance read Saslaw [16].

6.3 Astrophysical objects

There are two kinds of astronomical systems that can be described by statistical methods in combination with non-relativistic gravitation: systems where the "particles" are macroscopic objects like stars or galaxies, and systems where the "particles" are elementary particles. Examples of systems with macroscopic objects, and where relaxation may have occured, are globular clusters ($\approx 10^5$ stars), galactic or open clusters ($\approx 10^2$ stars), galactic nuclei ($\approx 10^8$ stars) and clusters of galaxies ($\approx 10^3$ galaxies). Examples of systems of elementary particles are the halos of dark matter particles that exist around galaxies.

Thirring [17] speculates about if the instability of a system with negative specific heat in contact with a heat bath (see chapter 6.4) can be used to describe formation of red giants, supernovae and the dense center of a galaxy.

6.4 Ensembles, negative specific heat and phase-transitions

From the different models of self-gravitating systems in chapter 7, we can conclude that often self-gravitating systems in three dimensions have negative specific heat. In any of these models, there are approximations, modifications or certain assumptions that more or less leaves the model with properties similar to real systems. The models are complements to each other, since they differ in different ways from real systems.

As will be discussed below, systems in the micro-canonical ensemble can have negative specific heat. Systems in the canonical ensemble can not have negative specific heat, as can be seen by (3.34).

Fig. 1 shows a system that has negative specific heat in one region of energy, B.

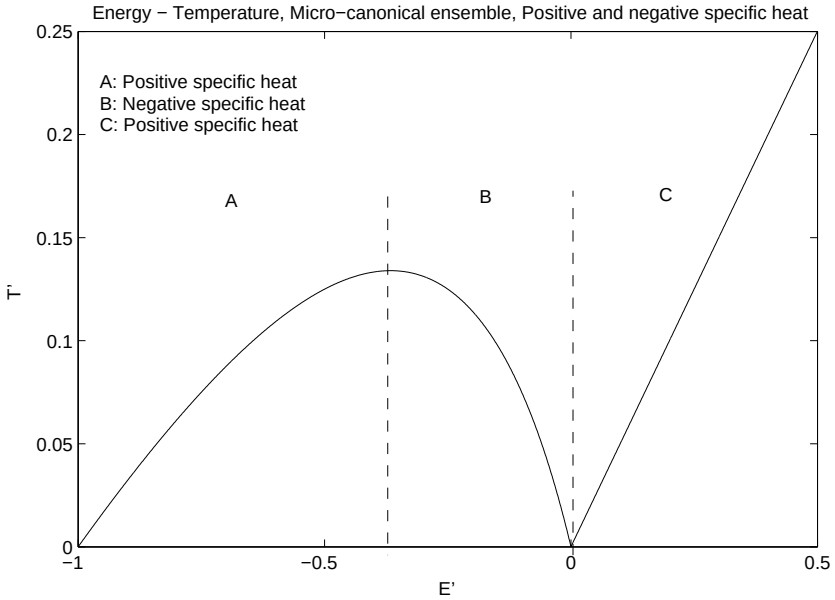

Fig. 1: Positive and negative specific heat. The curve is typical for 3-dimensional selfgravitating systems with inner and outer cutoff. This specific curve results from the binary star model, see chapter 7.4. The axes are rescaled to dimensionless units.

In region A, the system is affected by the pressure from a wall prohibiting the particles from coming close to the gravitating centre. In this region, energy supplied mainly increases the kinetic energy of the particle, and only to some extent their potential energy, and the temperature increases. In region B, the pressure on the walls is not significant, the energy supplied results in an increase of the potential energy that is bigger than the supplied energy, and the kinetic energy as well as the temperature decreases. In region C, the pressure from the outer wall prohibiting particles to escape from the system, is significant, and energy supplied can not increase the potential energy considerably, while the kinetic energy and so the temperature increases. The properties of the curve in fig. 1 is typical for three-dimensional self-gravitating systems in the micro-

canonical ensemble, even though the exact form of the curve may differ between different systems.

The canonical ensemble is a little bit troublesome to understand in the case of self-gravitating systems. It is possible to use the partition function, (3.20), on self-gravitating systems without asking any questions about its applicability. But, it may be interesting to investigate how this ensemble can be interpreted in physical terms in the case of a self-gravitating system. At first, let us look at two properties of systems with negative specific heat that are easy to derive and are listed for instance in Lynden-Bell [10]: 1. Two negative specific heat systems in thermal contact can not attain thermal equilibrium. 2. A negative specific heat system can attain thermal equilibrium with a positive specific heat system only if their total specific heat is negative. From these two properties we can conclude that a negative specific heat system can not be in thermal equilibrium with a heat bath (for a definition of heat bath see chapter 3), no matter if the heat bath has negative or positive specific heat. This makes systems in the canonical ensemble (with negative specific heat if studied in the micro-canonical ensemble) a hypothetical construction that is not theoretically consistent and can not exist in reality, if the system is not modified in some sense. We will discuss this below.

To understand systems in the canonical ensemble we must look at a phenomenon called "phase-transition". If the properties (for instance the density distribution or specific heat) of a system change dramatically under a very small change in for instance temperature, there is a "phase-transition". In this essay we study phase transitions that are associated with a specific heat that is very high in the region of temperature where the transition occurs. This means, the system undergoes a change in energy, with corresponding changes in configuration, while the change in temperature is very small. The phase-transition is associated with a discontinuity in the free energy of the system. The old "Ehrenfest scheme" is now regarded as inaccurate. The modern scheme says that in a first-order phase transition, there is "latent heat" in the system. Second-order phase-transistions, also called continuous phase-transitions, do not involve latent heat. At a first-order phase-transition the energy as a function of temperature is discontinous, i.e. there is a jump in energy at the temperature of the transition, (fig. 2). In a phase transition of the second kind, the corresponding function is continuous, (fig. 3). In the case of a phase transition of the first kind, the function (the dashed part is included) is multiple-valued before considering the Helmholtz free energy of the system, (3.26). Where there are multiple possibilities, it occurs the one that minimizes this quantity. The discontinuity occurs where the Helmholtz free energy of the two branches are equal.

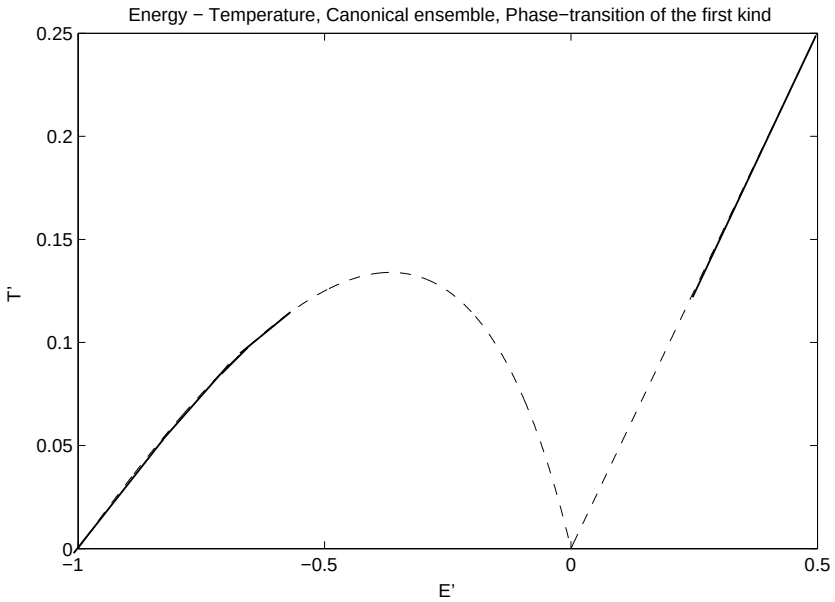

Fig. 2: Phase-transition of the first kind. The axes are rescaled to dimensionless units. The dashed curve represents the same system in the micro-canonical ensemble. The two curves have different temperature scales.

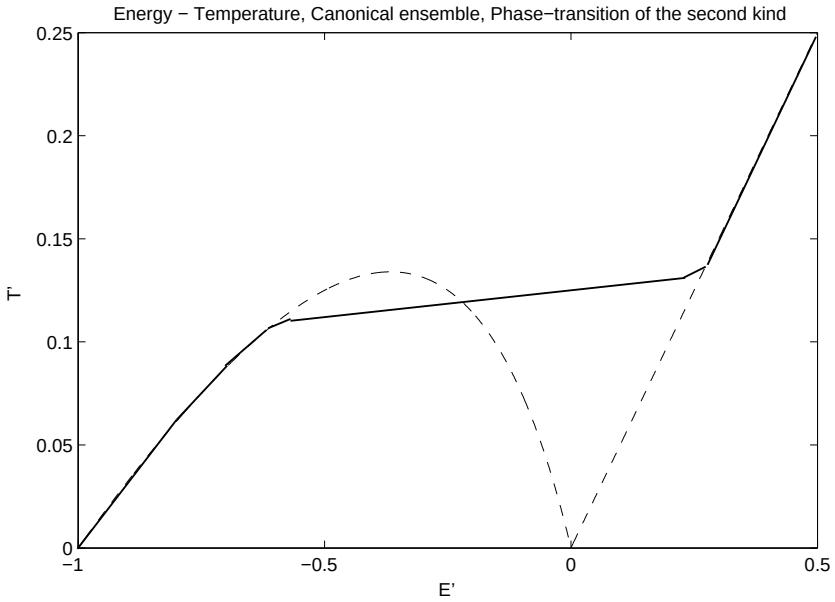

Fig. 3: Phase-transition of the second kind. The axes are rescaled to dimensionless units. The dashed curve represents the same system in the micro-canonical ensemble. The two curves have different temperature scales.

We will see in chapter 7, that it is characteristic for some simple models of self-gravitating systems to have negative specific heat in some region of energy in the micro- canonical ensemble, and a phase-transition with a huge amount of

positive specific heat in the corresponding region when studied in the canonical ensemble. See fig. 1, fig. 2 and fig. 3. This difference between the two ensembles is quite reasonable, as can be seen by (3.34). The fluctuation in energy for a system in the canonical ensemble goes to zero when the specific heat goes to zero, and the system behaves as it is in the micro-canonical ensemble. For high specific heat, the fluctuations are big, and the two ensembles behave very differently.

Often the same diagram over the relation between energy and temperature for a system can be used in both the micro-canonical and the canonical ensemble, if it is complemented with information about the Helmholtz free energy.

There is a way to make systems of negative specific heat stable in thermal contact with the surrounding. That is to introduce an inner and an outer cutoff that limits the potential energy of the system (for cutoffs, see chapter 6.6). In fig. 4 wee see the relation between energy and temperature for a system with negative specific heat in the micro-canonical ensemble (the system is isolated) with cutoffs. We then connect it very weakly to a heat bath so that there can be an energy flow from the hotter to the colder part, but with small energy fluctuations, so that the system still behaves like in the micro-canonical ensemble. The temperature of the heat bath is marked by a dotted line. A system initially at A gets energy from the heat bath during its trip along the curve. First, before the system reaches B, this increases the energy and lowers the temperature (because the specific heat is negative). Then, during the trip to C, the system still gains energy, but the temperature increases (because the specific heat is positive due to the outer cutoff). At C the system is in equilibrium with the heat bath. In principle the same process occurs if the system is initially at D. Then the sign of the specific heat changes at E, and stability is obtained at F due to the inner cutoff.

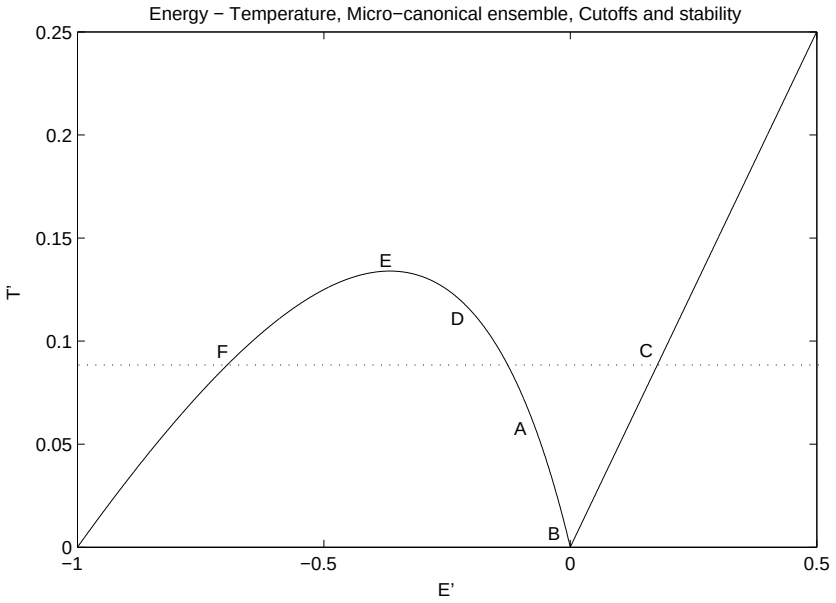

Fig. 4: A system in the micro-canonical ensemble with a (weak) thermal contact with the surrounding, can be made stable by introducing cutoffs. A system initially at A

becomes stable at C. A system initially at D becomes stable at F. The temperature of the surrounding heat bath is marked by a dotted line.

Sometimes it is useful to study a part of a self-gravitating system, a fixed volume that contains a fluctuating number of particles, N, and a fluctuating amount of energy, E. Since the surrounding part of the system serves as both a particle reservoir and a heat bath, the results for the grand canonical ensemble can be applied to this subsystem.

6.5 Local entropy maxima

A closed system adapts itself so that it attains maximum entropy under the constraints applied. This may result in, for instance, a specific density distribution, compatible with the constraints. If we let the different free (unconstrained) parameters in the system constitute a space, and there exists a point in this space where the entropy is higher than in any neighbouring points, then this point represents a "local maximum" for the entropy. The system will fluctuate around this point, when the system is in internal equilibrium. If we vary some constrained parameter of the system continously, the point with a local entropy maximum will move through the space. There may be some regions of the constrained parameters where no local maximum for the entropy exists in the space defined by the unconstrained parameters. That means, that it is then possible for the system to move through this space, and it will not reach a maximum in entropy while some unconstrained parameter goes to infinity. The system is then not stable. This phenomenon occurs, for instance, in the theory of "the gravo-thermal catastrophe". The latter phenomenon will be discussed in chapter 7.1.

6.6 Cutoffs

Let us look at an isolated system with energy E, where N particles, each with mass m, gravitate each other in a D-dimensional space. We look at the contribution to the entropy from the parts of the phase-space where particles are spatially close to each other. Our system has, according to (3.39), (with m_j = constant, $n_s = DN$ and $n_m = DN$), the phase-space volume

$$g(E) \propto \int \mathrm{d}x_{D1}^D \, \mathrm{d}x_{D2}^D \, \ldots \mathrm{d}x_{DN}^D \, (E + \frac{1}{2} \sum_{i \neq j} C_D m^2 |\vec{x}_{Di} - \vec{x}_{Dj}|^\nu)^{\frac{DN}{2}-1} \qquad (6.2)$$

We change integration variable

$$\vec{S}_D = \vec{x}_{D1} - \vec{x}_{D2} \qquad (6.3)$$

(6.2) and (6.3) give

$$g(E) \propto \int \mathrm{d}x_{D2}^D \, \mathrm{d}x_{D3}^D \, \ldots \mathrm{d}x_{DN}^D \, I(\vec{x}_{D2}, \ldots, \vec{x}_{DN}) \qquad (6.4)$$

where

$$I(\vec{x}_{D2},\ldots,\vec{x}_{DN}) =$$
$$= \int \mathrm{d}s^D \,(E + C_D m^2 |\vec{s}_D|^\nu + \sum_{j=3}^{N} C_D m^2 |\vec{s}_D + \vec{x}_{D2} - \vec{x}_{Dj}|^\nu +$$
$$+ \frac{1}{2} \sum_{i,j=2}^{N} C_D m^2 |\vec{x}_{Di} - \vec{x}_{Dj}|^\nu)^{\frac{DN}{2}-1} \tag{6.5}$$

This integral is divergent in some regions of D, ν and N. Near $|\vec{S}_D| = 0$ it behaves as

$$I_0 = \int_0^\epsilon \mathrm{d}s \; s^{D-1} (C_D m^2 s^\nu)^{\frac{DN}{2}-1} \propto lim_{\epsilon\to 0} \epsilon^{\frac{D\nu N}{2}+D-\nu} \tag{6.6}$$

For gravitation we have, according to (2.28), that $\nu = 2 - D$ $(D \geq 3)$. From (6.6) we then have for gravitation

$$I_0 \propto lim_{\epsilon\to 0}\, \epsilon^{ND(1-\frac{D}{2})+2(D-1)} \tag{6.7}$$

which is convergent only for

$$N \leq \frac{4(D-1)}{D(D-2)} \tag{6.8}$$

This implies that the phase-space of the system is infinite for $N > N_0$, where $N_0 = 2$ in three dimensions, $N_0 = 1$ in four or five dimensions, and $N_0 = 0$ for six dimensions or more. This results for $N > N_0$ in no local maxima for the entropy, and at least for $3 \leq D \leq 5$ some particles will come closer and closer to each other, without limit. Mathematically, this gives an infinite phase-space volume for the system. The solution to this, is to introduce a limiting sphere around every particle, that prohibits other particles from coming close. Another solution is to limit the potential at small distances. The common method is the former one. In models where there is no gravitation between the particles, but there is a centre of gravitation acting on all the particles, a sphere stopping particles to come close to the centre is sufficient. We call this kind of sphere, an inner limiting wall. Let us now look at the contribution to the phase-space volume of a system from one particle moving to infinity in spatial space. We assume that the system either (I) has spheres prohibiting particles from coming close to each other, or (II) a sphere that prohibits particles from coming close to a gravitating centre. The criteria for the possibility of a particle moving to infinity in space, is that the system's total energy, E_N, is related to the potential energy of the system when $N-1$ particles have the lowest possible kinetic energy (which means that in a system of type (I), $N-1$ particles are closely packed, or in a system of type (II), $N-1$ particles resides on the limiting sphere), $E_0(N-1)$, and the kinetic energy of the system, E_k, by

$$E_N > E_{0(N-1)} \tag{6.9}$$

This represents the situation when there is at least so much energy in the system so that there is a possibility that $N-1$ number of particles have the minimum

potential energy available, and the N:th particle can exist at infinity. Since the spatial volume at infinity is infinitely large, a system fulfilling (6.9) will have no local maximum in entropy. Physically, it means that particles will escape from the system, and we may successively redefine the system by reducing N and, eventually, E. The process will continue until (6.9) is no longer fulfilled. Not until then there will be a local maximum in entropy that represents the properties of a stable system. But then we may not have the kind of system we expected to study. The solution to these problems is to introduce an outer limiting sphere that restricts particles from going to infinity. Mathematically, this corresponds to a restriction of the volume of integration when calculating the phase-space volume.

6.7 Application of The Virial theorem

It is quite clear that there is a relation between the energy, potential energy and the kinetic energy of a self-gravitating system, (5.15). The Virial theorem tells there is a relation between the time averages of any two of these. Especially simple, see (5.14), (5.16) and (5.17), the relations are when no limiting wall (cutoff) affects the particles in the system. This occurs, for instance, when the energy of the system is not so high that the outer wall feels a significant pressure from the particles, and not so low that the inner wall feels a significant pressure. If the pressure on some wall is significant, we may use the form of the Virial theorem where there is a term containing the pressure and the volume of the system, see (5.19), (5.20) and (5.21).

We will now try to relate a spherical-symmetric, continuous and stable mass distribution with mass $M(r)$ inside spherical shells with radius r, to the time averaged total energy of the system, $< E >$. The time averaged potential energy, $< V >$, can be expressed without use of the Virial theorem.

$$\begin{aligned} < V > &= \frac{1}{2}\int \mathrm{d}M\,\phi(r) = \frac{1}{2}\int \mathrm{d}r\,\frac{\mathrm{d}M(r)}{\mathrm{d}r}\phi(r) = \\ &= \frac{2\pi G}{S_D}\int \mathrm{d}r\,\frac{\mathrm{d}M(r)}{\mathrm{d}r}\int_\infty^r \mathrm{d}r'\,\frac{M(r')}{r'^{D-1}} \end{aligned} \tag{6.10}$$

where ϕ is the gravitational field, and we have used the following expression for ϕ:

$$\phi(r) = -\int_\infty^r \mathrm{d}r'\,\frac{F(r')}{m} = \frac{4\pi G}{S_D}\int_\infty^r \mathrm{d}r'\,\frac{M(r')}{r'^{D-1}} \tag{6.11}$$

where we have used a formula similar to (2.30) for a spherical-symmetric mass distribution. $F(r')$ is the gravitational force on a particle with mass m at radial position r'. Now, we will calculate the time averaged kinetic energy, $< K >$. Let us call the mass of a particle m, the force on a particle $\vec{F}$, and the particle's position $\vec{r}$. Let us call the mass density $\rho(r)$. (6.11) gives

$$S_D r^{D-1} F = -4\pi G M(r) m \tag{6.12}$$

and from this, we have for one term in the Virial of Clausius

$$f(r) = \vec{F}\cdot\vec{r} = -\frac{4\pi G m M(r)}{S_D r^{D-2}} \tag{6.13}$$

The Virial theorem, (6.12), (6.13) and $S_D r^{D-1}\rho(r) = \frac{\mathrm{d}M(r)}{\mathrm{d}r}$ give

$$< K >= -\frac{1}{2} < \sum_i \vec{F}_i \cdot \vec{r}_i >= -\frac{1}{2}\int \frac{\mathrm{d}M}{m} f(r) = -\frac{1}{2}\int \frac{\mathrm{d}V\,\rho(r)}{m} f(r) =$$
$$= -\frac{1}{2}\int \frac{S_D r^{D-1}\mathrm{d}r\,\rho(r)}{m}(-\frac{4\pi G m M(r)}{S_D r^{D-2}}) = 2\pi G\int \mathrm{d}r\, r\rho(r)M(r) =$$
$$= \frac{2\pi G}{S_D}\int \mathrm{d}r\,\frac{1}{r^{D-2}}\frac{\mathrm{d}M(r)}{\mathrm{d}r}M(r) \qquad (6.14)$$

(6.10) and (6.14) give for the time averaged total energy

$$< E >=< K > + < V >= \frac{2\pi G}{S_D}\int \mathrm{d}r\,\frac{\mathrm{d}M(r)}{\mathrm{d}r}(\frac{M(r)}{r^{D-2}} + \int_\infty^r \mathrm{d}r'\,\frac{M(r')}{r'^{D-1}}) \qquad (6.15)$$

Observe that the mass distribution $M(r)$ can not be chosen arbitrarily, since the application of the Virial theorem demands a stable system, and this is true only for some $M(r)$, for instance those which are associated with an isothermal system, see chapter 6.8.

Let us now try to represent the Virial theorem in another form, for particles interacting with each other with an ordinary gravitational (or Coloumb) force. Let us denote the force on particle i with $\vec{F}_i$, the force on particle i due to the field from particle j with $\vec{F}_{ij}$, the position of particle i with $\vec{R}_i$, the position of particle i relative to particle j with $\vec{r}_{ij}$, the angle between $\vec{R}_j$ and $\vec{r}_{ij}$ with θ_{ij}, the potential energy of particle i due to the field from particle j with V_{ij}, the charge (positive or negative) of particle i with m_i, and the force constant with G. Then we have

$$< K >= -\frac{1}{2} < \sum_i \vec{F}_i \cdot \vec{R}_i >= -\frac{1}{2} < \sum_i \sum_{j\neq i} \vec{F}_{ij} \cdot (\vec{r}_{ij} + \vec{R}_j) >=$$
$$= -\frac{1}{2} < \sum_i \sum_{j\neq i} (-\frac{G m_j m_i}{r_{ij}^2})(r_{ij} + R_j\cos(\theta_{ij})) >=$$
$$= -\frac{1}{2} < \sum_i \sum_{j\neq i} V_{ij} > -\frac{1}{2} < \sum_i \sum_{j\neq i} V_{ij}\frac{R_j}{r_{ij}}\cos(\theta_{ij}) >=$$
$$= - < V > -\frac{1}{2}\sum_i \sum_{j\neq i} < V_{ij}\frac{R_j}{r_{ij}}\cos(\theta_{ij}) > \qquad (6.16)$$

For the case of two particles, with origin at the centre of mass, where $\frac{R_1}{r_{21}} = \gamma$ $(0 < \gamma < 1)$, we have $\frac{R_2}{r_{12}} = 1 - \gamma$ and $\theta_{21} = \theta_{12} = \pi$, and since $< V >= \frac{1}{2}(V_{12} + V_{21})$ and $V_{21} = V_{12}$, we obtain (5.14). Even for cases with more particles than two, there are symmetries so that, despite the complicated form of the expression in the summation signs, the system agrees with (5.14).

Thirring [17] makes a discussion about negative specific heat of matter. Thirring discusses whether the Virial theorem, in the form (5.14), can be used to prove that all condensed matter have negative specific heat, but he finds reasons, for instance the need of cutoffs, for that this is not true.

6.8 The mean field approximation

Here, we will derive the so called mean field limit of a self-gravitating system in three dimensions, in which we ignore the granularity and the fluctuations present in an N-particle system. The approximation is widely used. This derivation is presented in Padmanabhan [15]. The spherical- symmetric case under this approximation is called "the isothermal sphere".

We start with a system of N particles confined in a volume, V, and interacting through the two-body potential $U(\vec{x}_i, \vec{x}_j)$. In the micro-canonical ensemble, there is for the entropy, S

$$\exp S = g(E) = \frac{1}{N!} \int \mathrm{d}^{3N}x \, \mathrm{d}^{3N}p \, \delta(E - H) = \\ = \frac{A}{N!} \int \mathrm{d}^{3N}x \, (E - \frac{1}{2} \sum_{i \neq j} U(\vec{x}_i, \vec{x}_j))^{\frac{3N}{2}} \tag{6.17}$$

where A is a constant, and we have approximated $\frac{3N}{2} - 1$ with $\frac{3N}{2}$. We divide V in M cells of equal size, large enough to contain many particles but small enough for the potential to be approximately constant within the cell. There is a standard result that

$$\frac{\mathrm{d}^{3N}x}{N!} = \sum_{n_1=1}^{\infty} \frac{1}{n_1!} \sum_{n_2=1}^{\infty} \frac{1}{n_2!} \cdots \sum_{n_M=1}^{\infty} \frac{1}{n_M!} \delta(N - \sum_a n_a) (\frac{V}{M})^N \tag{6.18}$$

Using (6.18) on (6.17), approximating the factorials with Stirling's formula (3.6), and omitting the unimportant constant A, give

$$\exp S \approx \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \cdots \sum_{n_M=1}^{\infty} \delta(N - \sum_a n_a) \exp S(\{n_a\}) \tag{6.19}$$

where

$$\exp S(\{n_a\}) = \frac{3}{2} N \ln(E - \frac{1}{2} \sum_{a \neq b}^{M} n_a U(\vec{x}_a, \vec{x}_b) n_b) - \sum_{a=1}^{M} n_a \ln(\frac{n_a M}{V}) \tag{6.20}$$

Next, we approximate by keeping only the term in (6.19) that is the biggest one. That is

$$\sum_{\{n_a\}} \exp S(\{n_a\}) \approx \exp S(\{n_{a,max}\}) \tag{6.21}$$

where $n_{a,max}$ is the solution to the variational problem

$$(\frac{\delta S}{\delta n_a})_{n_a = n_{a,max}} = 0 \qquad \sum_{a=1}^{M} n_a = N \tag{6.22}$$

(6.19) and (6.22) give

$$\frac{1}{T} \sum_{b=1}^{M} U(\vec{x}_a, \vec{x}_b) n_{b,max} + \ln(n_{a,max} \frac{M}{V}) = \text{constant} \tag{6.23}$$

where we defined the temperature, T, as

$$\frac{1}{kT} = \frac{3}{2}N(E - \frac{1}{2}\sum_{a\neq b}^{M} n_a U(\vec{x}_a, \vec{x}_b)n_b)^{-1} \tag{6.24}$$

We see from (6.20) that T is the correct temperature, according to the definition, (3.15). We go to the continuum limit by the replacements

$$\begin{cases} n_{a,max}\frac{M}{V} = \rho(\vec{x}_a) \\ \sum_{a=1}^{M} \to \frac{M}{V} \end{cases} \tag{6.25}$$

In this limit, the extremum solution, (6.23), becomes

$$\begin{cases} \rho(\vec{x}) = C\exp\left(-\beta\phi(\vec{x})\right) \\ \phi(\vec{x}) = \int \mathrm{d}^3 y\, U(\vec{x},\vec{y})\rho(\vec{y}) \end{cases} \tag{6.26}$$

In the case of three-dimensional gravitation, we have

$$\begin{cases} \rho(\vec{x}) = C\exp\left(-\beta\phi(\vec{x})\right) \\ \phi(\vec{x}) = -G\int \mathrm{d}^3 y\, \frac{\rho(\vec{y})}{|\vec{x}-\vec{y}|} \end{cases} \tag{6.27}$$

In the derivation of (6.27), we searched for an extremum of the system's entropy. This does not ensure a maximum of the entropy. In some regions of the parameters of the system, this extremum corresponds to a saddle point for the entropy. Only in some regions, there is a maximum. This will be discussed in chapter 7.1, and is related to a phenomena called "the gravo-thermal catastrophe".

Chapter 7

Special models of self-gravitating systems

The models presented here have been constructed to shed light on properties of real self-gravitating systems. These models rely on approximations or modifications of real systems. This is to allow mathematical analysis, which is very hard to do on the exact real system. The different models complement each other, since the approximations and modifications are different in the different models.

7.1 The isothermal sphere

This is one of the the most widely used models to analyse properties of real astrophysical systems. Details about this model and the associated phenomena "the gravo-thermal catastrophe" can be found in Lynden-Bell/Wood [12] and Padmanabhan [15]. Antonov [1] showed that, under the mean field approximation, the spherical- symmetric case is the case with highest entropy. This case is called the "isothermal sphere". Antonov [1] also studied exact criterias for the stability of such a system. The system has mass M and energy E (negative), and is confined within a spherical shell of radius R. Let us call the density in the centre of the sphere ρ_0, and the density at the distance R from the centre with ρ_R. The system is stable (has a local entropy maxima) if $\frac{RE}{GM^2} > -0.335$ and $\frac{\rho_c}{\rho_R} < 709$. If $\frac{RE}{GM^2} > -0.335$ and $\frac{\rho_c}{\rho_R} > 709$ the system is in a metastable state where the entropy extremum is a saddle point. The system is isothermal, but evolves towards higher and higher temperature and ρ_0. If $\frac{RE}{GM^2} < -0.335$ the entropy has no extremum, and the system is not isothermal. In the two latter cases, a central nucleus with higher and higher temperature evolves. This is what is ment with "the gravo-thermal catastrophe".

7.2 The Virial model

Let us look at the possibility of using the Virial theorem to investigate the specific heat of a system. We regard the case where the pressure on an eventual limiting wall not is significant, and they attract each other with potentials on

the form $\phi = C_D r^\nu$, where C_D and ν have the same sign. (5.16) reads

$$< E >= \frac{\nu + 2}{\nu} < K > \tag{7.1}$$

(3.29) and (7.1) give

$$C_V = \frac{\nu + 2}{\nu} \frac{\mathrm{d}< K >}{\mathrm{d}T} \tag{7.2}$$

For many systems, $< K >$ is a monotonely increasing function of the temperature. To show that this is the case for any self-gravitating system may not be easy, but we investigate the case where the relation is assumed to hold. Then, we have

$$\frac{\mathrm{d}< K >}{\mathrm{d}T} > 0 \tag{7.3}$$

(7.2) and (7.3) implies that C_V is negative exactly when $\nu = -1$. For gravitation we have (2.28), and we get

$$C_V = \frac{4 - D}{2 - D} \frac{\mathrm{d}< K >}{\mathrm{d}T} \qquad D \geq 3 \tag{7.4}$$

We then have from (7.4) and (7.3) that C_V is negative in three dimensions, zero in four dimensions, and positive in five dimensions or more.

7.3 The circular orbit model

Let us now study a model that gives similar results, but is based on assumptions that hold for a single particle in a circular orbit in a central force potential on the form $\phi = C_D r^\nu$ ($\nu \neq 0$), where C_D and ν have the same sign. Below, we will see that the particle has to be confined between two spherical walls. This model lacks a physical mechanism that makes the system ergodic. The ergodicity has to be assumed explicitly. It may also be possible to add a mechanism that makes the system ergodic. We call this model the "circular orbit model". In this calculation, we use the sign "$\propto$" in the sense "proportional to". (The two expressions only differ by a multiplicative constant independent of E, r and $\vec{p}$). First, we formulate the equations that define the properties of the system. The potential energy of the particle, V, is

$$V = mC_D r^\nu \tag{7.5}$$

where r is the radius of the orbit, and m the mass of the particle. It is easy to derive that the kinetic energy of the particle, K, is

$$K = \frac{\nu}{2} mC_D r^\nu \tag{7.6}$$

The total energy of the particle, E, is

$$E = V + K = mC_D(1 + \frac{\nu}{2})r^\nu \propto r^\nu \tag{7.7}$$

We assume that the particle is confined spatially within a distance r to $r + \mathrm{d}r'$ from the centre, which gives for the spatial volume of the system

$$V_r \propto r^{D-1}\mathrm{d}r' \tag{7.8}$$

where D ($D \geq 3$) is the number of dimensions of the space. We assume the length of the linear momentum of the particle to be in the interval $|\vec{p}|$ to $|\vec{p}|+\mathrm{d}|\vec{p}|$. $\vec{p}$ is directed perpendicular to the central force potential's radius, so that the particle moves at the surface of a sphere in a D-dimensional space. This surface has $D-1$ dimensions. Except for a length in an interval of size $\mathrm{d}|\vec{p}|$, the direction of $\vec{p}$ is arbitrary as long as it is a tangent to the $(D-1)$-dimensional surface. The change of direction of $\vec{p}$ is then performed in $D-2$ dimensions, and $|\vec{p}|$ is the radius defining the size of this $(D-2)$-dimensional space. For the size of the momentum-space, V_p, we then have

$$V_p \propto |\vec{p}|^{D-2} \frac{\mathrm{d}|\vec{p}|}{\mathrm{d}r} \mathrm{d}r' \tag{7.9}$$

Provided that the particle is in V_r and in V_p and $\mathrm{d}r'$ is infinitesimal, the spatial coordinates and the phase coordinates are uncorrelated, which can be expressed

$$V_{rp} = V_r V_p \tag{7.10}$$

where V_{rp} is the volume of the system's phase-space. Using some common relations from classical physics and (7.6), we get

$$|\vec{p}|^2 = 2mK = m^2 \nu C_D r^\nu \propto r^\nu \tag{7.11}$$

From (7.11) we get

$$\frac{\mathrm{d}|\vec{p}|}{\mathrm{d}r} \propto r^{\frac{\nu}{2}-1} \tag{7.12}$$

(7.8), (7.9), (7.10), (7.11) and (7.12) give

$$V_{rp} \propto r^{D-1+\frac{\nu(D-2)}{2}+\frac{\nu}{2}-1} \mathrm{d}r'^2 \tag{7.13}$$

We need to express $\mathrm{d}r'$ in terms of the coordinate r. The properties of a system can, often, be formulated as a function of a dimensionless parameter on for instance the form $\frac{GMm}{a^{-\nu}E}$, where G is the gravitational constant, M and m masses, a a distance, and E the energy of the system. The relevant aspect of this is here that when distances appear in statistical models of self-gravitating systems, they are raised to an integer and multiplied by the energy of the system. This means, that if we scale the energy of the system, we can keep the properties of the system by a compensating scaling of the space coordinates in the model. (In a Hamiltonian, also the time would be scaled.) In Landau/Lifshitz [9], the concept of "mechanical similarity" is defined, which considers relations between scalings of parameters in a Lagrangian. Let us, in this essay, call scalings that leaves the phase-space volume the same (except for a multiplicative constant) for "statisical similarity" For instance some of the "special models" in chapter 7 possesses statistical similarity. This clearly regards the "binary star" model, see chapter 7.4. We may think of our present model as having two walls confinig the particle: one wall at radius r, and one at radius $r + \mathrm{d}r'$. If our model shall possess statistical similarity, it is natural to think that $\mathrm{d}r'$ has to be proportional to r. So, we have

$$\mathrm{d}r' \propto r \tag{7.14}$$

By use of (7.14), we can rewrite (7.13) as

$$V_{rp} \propto r^{D-1+\frac{D\nu}{2}-\frac{\nu}{2}} \tag{7.15}$$

We get the temperature of the system by using (3.15), (3.1), (7.15) and (7.7).

$$T = \frac{1}{k}\frac{E}{\frac{D-1}{\nu} + \frac{D}{2} - \frac{1}{2}} \tag{7.16}$$

(3.28) and (7.16) give

$$C_V = k(\frac{D-1}{\nu} + \frac{D}{2} - \frac{1}{2}) \tag{7.17}$$

We see that the specific heat is negative exactly when $\nu = -1$. Gravitation obeys (2.28), which together with (7.17) give

$$C_V = k(\frac{D-1}{2-D} + \frac{D}{2} - \frac{1}{2}) \tag{7.18}$$

We see that the specific heat of the system is negative in three dimensions, zero in four dimensions, and positive in five dimensions or more. This result is similar to that obtained by using the Virial theorem and the assumption that $K(T)$ is a monotonely increasing function, see chapter (7.2). In our circular orbit model, the most doubtful assumption is the one of statistical similarity and its application to this problem. But without this assumption, we would not get reasonable results for the specific heat in different dimensions. If we for instance would set $\mathrm{d}r' \propto$ constant, then we get $C_V = 0$ in three dimensions. Statistical similarity is an important property of self-gravitating systems. The things in real systems that may differ from the premises in the circular orbit model and may limit the applicability of the results of this model, is that in other models the particles may be allowed to not move in circular orbits, they may distribute the energy between each other, and they may interact with each other.

7.4 The binary star model

The binary star model is rather easy to analyse, since it contains only two particles, each with mass m, that attract each other. The model is presented for instance in Padmanabhan [15]. In this essay, we generalise the model to D dimensions ($D \geq 3$) and a potential, $\phi = C_D r^\nu$, with arbitrary ν ($\nu \neq 0$), where C_D and ν have the same sign. The Hamiltonian is

$$H(\vec{p}_0, \vec{q}_0, \vec{p}_1, \vec{q}_1) = \frac{p_0^2}{2m_0} + \frac{p_1^2}{2m_1} + mC_D r^\nu \tag{7.19}$$

where m_0 is the two particle's total mass ($= 2m$), m_1 their reduced mass ($= \frac{m}{2}$), $\vec{p}_0$ the conjugate linear momentum of their centre of mass, $\vec{p}_1$ their relative conjugate linear momentum, and r the distance between the two particles ($= q_1$). $\vec{q}_0$ is the position of the centre of mass. The particles are restricted to the volume of a sphere with much larger diameter than the maximum distance between the particles (because the distance between the particles we want to be limited by the energy of the system only, see below). This sphere serves to make the system ergodic.Since the system has momentum degrees of freedom corresponding to two particles, the volume of the phase-space is, according to (3.40), (with $m_j =$ constant, $n_{sr} = D-1$ and $n_m = 2D$)

$$g^{(0)}(E) = C\int_a^{r_{max}} \mathrm{d}r\, r^{D-1}(E - mC_D r^\nu)^{D-1} \tag{7.20}$$

where we, for convenience, have included dE within the constant C. We have in (7.20) introduced cutoffs, limits on r: an inner sphere with radius a, and an outer sphere with radius R. The outer sphere affects the system if $E > mC_D R^\nu$. Then $r_{max} = R$. If $E < mC_D R^\nu$, then $r_{max} = (\frac{E}{mC_D})^{\frac{1}{\nu}}$, which represents zero kinetic energy (the expression in parenthesis in (7.20) is zero for $r = r_{max}$). In this case, the energy of the system sets the upper limit on r. For gravitation in three dimensions, we have $D = 3$, (2.12) ($C_D = -Gm$) and (2.28) ($\nu = 2 - D$), and then (7.20) gives

$$g_3^{(0)}(E) = \begin{cases} C\frac{(Gm^2)^3}{3}\frac{(1+\frac{aE}{Gm^2})^3}{E} & -\frac{Gm^2}{a} < E < -\frac{Gm^2}{R} \\ C\frac{(Gm^2)^3}{3}\frac{(1+\frac{RE}{Gm^2})^3-(1+\frac{aE}{Gm^2})^3}{E} & -\frac{Gm^2}{R} < E < \infty \end{cases} \tag{7.21}$$

We use (3.15) and (3.1) on (7.21) to calculate the temperature, T, as a function of the energy, E. This function is caracteristic for self-gravitating systems in three dimensions and with cutoff, see fig. 5.

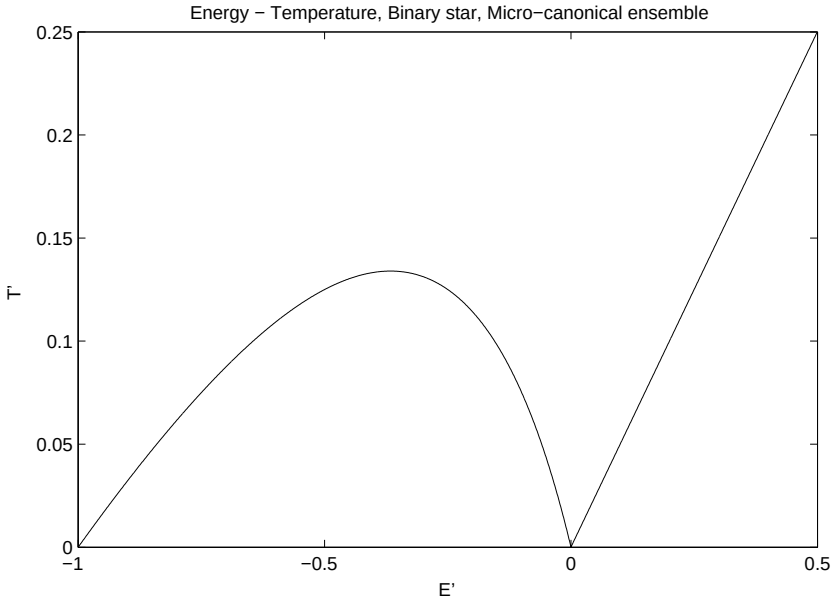

Fig. 5: Binary star model. The axes are rescaled to dimensionless units, $E' = \frac{aE}{Gm^2}$ and $T' = \frac{akT}{Gm^2}$, where k is Boltzmann's constant. The cutoffs are set to $\frac{R}{a} = 5 \cdot 10^3$.

Now, we calculate the specific heat, C_V, in the general case. To calculate the specific heat according to (3.41), we need to know $g^{(1)}(E)(\equiv g'(E))$ and $g^{(2)}(E)(\equiv g''(E))$ as well as $g^{(0)}(E)(\equiv g(E))$.

$$g^{(1)}(E) = C(D-1)\int_a^{r_{max}} \mathrm{d}r\, r^{D-1}(E - mC_D r^\nu)^{D-2} \tag{7.22}$$

$$g^{(2)}(E) = C(D-1)(D-2)\int_a^{r_{max}} \mathrm{d}r\, r^{D-1}(E - mC_D r^\nu)^{D-3} \tag{7.23}$$

To solve the integrals, we use partial integration.

$$\int_a^{r_{max}} \mathrm{d}r\, r^{D-1}(E - mC_D r^\nu)^{n_j} = \frac{1}{D}[r^D(E - mC_D r^\nu)^{n_j}]_a^{r_{max}} +$$

$$+\frac{n_j\nu m C_D}{D(D+\nu)}[r^{D+\nu}(E-mC_Dr^\nu)^{n_j-1}]_a^{r_{max}}+$$
$$+\frac{n_j(n_j-1)\nu^2m^2C_D^2}{D(D+\nu)(D+2\nu)}[r^{D+2\nu}(E-mC_Dr^\nu)^{n_j-2}]_a^{r_{max}}+\ldots$$
$$\ldots+\frac{n_j(n_j-1)\cdot\ldots\cdot1\cdot\nu^{n_j}m^{n_j}C_D^{n_j}}{D(D+\nu)(D+2\nu)\cdot\ldots\cdot(D+n_j\nu)}[r^{D+n_j\nu}]_a^{r_{max}} \tag{7.24}$$

where n_j refers to the corresponding number in the formula for $g^{(j)}(E)$. This expression holds for cases where no denominator contains a zero factor, that is, when

$$D+i\nu\neq0 \qquad i=0,1,\ldots,n_j \tag{7.25}$$

We are interested in cases that may involve negative specific heat, that is, regions corresponding to region B in fig. 1. This implies that the energy sets a limit on r, that is $r_{max}=(\frac{E}{mC_D})^{\frac{1}{\nu}}$. ($E$ has the same sign as ν, since C_D has the same sign as ν.) All terms in (7.24) that contains r_{max}, except for the last one, then becomes zero. Except for one r_{max} term, only terms containing powers of a remains. The sign on the specific heat in a region like region B in fig. 1 do not alter if we let $a\to0$, and since we are not interested in regions corresponding to region A in fig. 1, where we always have positive specific heat, the condition $a\to0$ can be used here. This simplifies the calculations. $a\to0$ can be regarded as representing an inner cutoff that really exists, but whose radius can be chosen arbitrarily close to zero.

We want to let $a\to0$ in our calculations, while E shall be regarded as finite. To be able to do that, despite the fact that a together with other parameters defines the system and that we shall be able to vary E, we have to regard the definition of the system valid only for $|E|<A$, where A is some (eventually very big) constant. When we have set this A, we can define our system by chosing an a arbitralily close to zero, so that for any E fulfilling $|E|<A$, small changes in a do not significantly affect the specific heat, that is, we have the limit $a\to0$ while E is finite.

Since (3.41) reads

$$C_V=k\frac{g^{(1)}(E)^2}{g^{(1)}(E)^2-g^{(0)}(E)g^{(2)}(E)} \tag{7.26}$$

it is possible to multiplicate $g^{(1)}(E)^2$ and $g^{(0)}(E)g^{(2)}(E)$ with one and the same factor without affecting C_V. More general, since $n_1=D-2$ (for $g^{(1)}(E)$), $n_0=D-1$ (for $g^{(0)}(E)$) and $n_2=D-3$ (for $g^{(2)}(E)$), multiplication of $g^{(j)}(E)$ with a factor $\gamma_1^{\gamma_2+\gamma_3n_j}$ (γ_i arbitrary, but not dependent of j) do not change C_V. Since $a\to0$, we can use this to simplify (7.24) by multiplying with a^{-l} where l is the lowest power of a in the terms in (7.24). Then the terms with a^l is transformed to terms constant in a, and all other terms goes to zero when $a\to0$.

First, we consider $D+n_j\nu>0$ for all j. We call this case A. Then the lowest power of a in (7.24) is 0. It appears in the last term, a term containing $r_{max}^{D+n_j\nu}$. The term in (7.24) that remain when $a\to0$, is this term. We also multiply with

$\frac{D(D+\nu)\cdot\ldots\cdot(D+(D-3)\nu)}{C(D-1)(D-2)\cdot\ldots\cdot 2(\nu m C_D)^{n_j}} r_{max}^{-(D+n_j\nu)}$. We then have

$$\begin{cases} g^{(0)} \sim \frac{1}{(D+(D-2)\nu)(D+(D-1)\nu)} \\ g^{(1)} \sim \frac{1}{D+(D-2)\nu} \\ g^{(2)} \sim 1 \end{cases} \tag{7.27}$$

where the "$\sim$" sign stands for equality after multiplication with an allowed factor and after taking the limit of a. For the specific heat, we have

$$C_V = k\frac{D+(D-1)\nu}{\nu} \qquad D+n_j\nu > 0 \tag{7.28}$$

This expression is positive for positive ν, and for negative ν negative in the complete interval where it is valid, namely for $\nu = -1$.

We then consider $D + n_j\nu < 0$ for all j. We call this case B. The terms that remain in (7.24) after multiplication with $a^{-(D+\nu n_j)}$ and when $a \to 0$ are the ones that contained $a^{D+\nu n_j}$, that is, one term from every integration bracket. We also multiply with $C^{-1}(mC_D)^{-n_j}$. From (7.20), (7.22), (7.23) (7.24) and (7.26) we then have

$$\begin{cases} g^{(0)} \sim \frac{1}{D} + \frac{(D-1)\nu}{D(D+\nu)} + \frac{(D-1)(D-2)\nu^2}{D(D+\nu)(D+2\nu)} + \ldots + \frac{(D-1)(D-2)\cdot\ldots\cdot 1\cdot\nu^{D-1}}{D(D+\nu)\cdot\ldots\cdot(D+(D-1)\nu)} \\ g^{(1)} \sim (D-1)(\frac{1}{D} + \frac{(D-2)\nu}{D(D+\nu)} + \frac{(D-2)(D-3)\nu^2}{D(D+\nu)(D+2\nu)} + \ldots + \frac{(D-2)(D-3)\cdot\ldots\cdot 1\cdot\nu^{D-2}}{D(D+\nu)\cdot\ldots\cdot(D+(D-2)\nu)}) \\ g^{(2)} \sim (D-2)(D-1)(\frac{1}{D} + \frac{(D-3)\nu}{D(D+\nu)} + \frac{(D-3)(D-4)\nu^2}{D(D+\nu)(D+2\nu)} + \ldots \\ \qquad \ldots + \frac{(D-3)(D-4)\cdot\ldots\cdot 1\cdot\nu^{D-3}}{D(D+\nu)\cdot\ldots\cdot(D+(D-3)\nu)}) \end{cases} \tag{7.29}$$

The formula (7.24) is valid for a certain j only for cases where (7.25) is fulfilled for this j. The two cases that we calculated on above, demands that (7.25) is fulfilled for all j. Now, we will calculate on the case where there is one i $(1 \leq i \leq D-3)$ for which (7.25) is not fulfilled for any j We call this case C. Then, during the process of successive partial integrations, when the factor $D + (i-1)\nu$ appears in the denominator of a term, the next integration will result in a term on the form $C_1 \ln r$, where C_1 is a constant in r. When all the steps of integration have been performed, one term will be logarithmic, and this term will have the form $C_2 r^{\nu(n_j - i)} \ln r$, and will be the important one when we let $a \to 0$. When we have done all the allowed reductions of factors in the $g^{(j)}$:s, we obtain

$$\begin{cases} g^{(0)} \sim 1 \\ g^{(1)} \sim D - i - 1 \\ g^{(2)} \sim (D-i-1)(D-i-2) \end{cases} \tag{7.30}$$

For the specific heat, we have

$$C_V = k(D-i-1) \qquad \exists i : D + i\nu = 0 \qquad 1 \leq i \leq D-3 \tag{7.31}$$

This expression is positive for all cases where it is valid.

Above, we have defined three categories to which the $g^{(j)}$:s can belong. Now, we investigate the cases for which all the $g^{(j)}$:s do not belong to the same category. We call this case D. When $a \to 0$, the most important terms in (7.24) are the ones with a logarithmic behavior in a, that means one term in every $g^{(j)}$ belonging to category C. The next most important kind of term is the one

with a polynomial dependence of a and with a negative exponent, that means terms in $g^{(j)}$:s belonging to category B. The least important kind of term is a one that is constant in a, namely the term remaining in $g^{(j)}$ when $a \to 0$ and $g^{(j)}$ belongs to category A. From this information we can conclude what $g^{(j)}$:s that dominates in (7.26) when calculating the specific heat. We then also use the following scheme, that is derived from (7.26):

$$\begin{cases} g^{(0)} \text{ most important } \Rightarrow C_V = 0 \\ g^{(1)} \text{ most important } \Rightarrow C_V = k \\ g^{(2)} \text{ most important } \Rightarrow C_V = 0 \\ g^{(0)} \text{ and } g^{(1)} \text{ most important } \Rightarrow C_V = k \\ g^{(0)} \text{ and } g^{(2)} \text{ most important } \Rightarrow C_V = 0 \\ g^{(1)} \text{ and } g^{(2)} \text{ most important } \Rightarrow C_V = k \\ g^{(0)} \text{ and } g^{(1)} \text{ and } g^{(2)} \text{ equally important } \Rightarrow \\ \qquad \Rightarrow C_V \text{ can be calculated according to cases A, B or C above} \end{cases} \tag{7.32}$$

In the table below, numerical values on C_V in terms of units of Boltzmann's constant, k, are presented. The type of case is also given.

ν	D							
	3	4	5	6	7	8	9	10
3	3.0 (A)	4.3 (A)	5.7 (A)	7.0 (A)	8.3 (A)	9.7 (A)	11 (A)	12 (A)
2	3.5 (A)	5.0 (A)	6.5 (A)	8.0 (A)	9.5 (A)	11 (A)	12 (A)	14 (A)
1	5.0 (A)	7.0 (A)	9.0 (A)	11 (A)	13 (A)	15 (A)	17 (A)	19 (A)
0	X	X	X	X	X	X	X	X
-1	-1 (A)	-1 (A)	-1 (A)	-1 (A)	-1 (A)	-1 (A)	-1 (A)	-1 (A)
-2	0 (D)	1 (D)	1 (D)	2 (C)	2.5 (B)	3 (C)	3.5 (B)	4 (C)
-3	1 (D)	1 (D)	2.3 (B)	3 (C)	3.7 (B)	4.3 (B)	5 (C)	5.7 (B)
-4	1 (D)	2 (C)	2.7 (B)	3.5 (B)	4.2 (B)	5 (C)	5.8 (B)	6.5 (B)
-5	1 (D)	2.3 (B)	3 (C)	3.8 (B)	4.6 (B)	5.4 (B)	6.2 (B)	7 (C)
-6	1 (D)	2.4 (B)	3.2 (B)	4 (C)	4.8 (B)	5.7 (B)	6.5 (B)	7.3 (B)

7.5 The Lynden-Bell/Lynden-Bell model

D. Lynden-Bell and R.M. Lynden-Bell, see [11], have constructed a model of a self-gravitating system that includes a large amount of particles, but can be exactly analysed. The particles are, in three dimensions, confined to a spherical surface of radius r. When it comes to the gravitational interaction, the mass of the particles is assumed to be uniformly distributed over the surface. The radius of the sphere is fluctuating as a result of fluctuations in the distribution between potential and kinetic energy in the system. The model is, in this essay, generalised to any number of dimensions, D (≥ 3), by assuming the sphere to have one dimension less than the space has. We will also let the parameter ν in the potential, $\phi = C_D r^\nu$, be arbitrary ($\neq 0$). C_D and ν have the same sign. Let us study a particle that moves on the sphere. In three dimensions the particle's spherical coordinates, r, θ_1 and θ_2 is related to the cartesian coordinates x_1, x_2 and x_3 by

$$\begin{cases} x_1 = r\cos\theta_1 \sin\theta_2 \\ x_2 = r\sin\theta_1 \sin\theta_2 \\ x_3 = r\cos\theta_1 \end{cases} \tag{7.33}$$

First, we suppose r constant, and study the case of time-dependent θ_i:s. The kinetic energy of the particle is

$$E_{kp} = \frac{m}{2}(\frac{\mathrm{d}x_1}{\mathrm{d}t})^2 + \frac{m}{2}(\frac{\mathrm{d}x_2}{\mathrm{d}t})^2 + \frac{m}{2}(\frac{\mathrm{d}x_3}{\mathrm{d}t})^2 = \frac{C_1^2 r^2 \dot{\theta_1}^2}{2} + \frac{C_2^2 r^2 \dot{\theta_2}^2}{2} \tag{7.34}$$

where the C_i:s are constants, independent of r, $\dot{r}$ and the $\dot{\theta}_i$:s, but dependent of the θ_i:s. In four dimensions we have

$$\begin{cases} x_1 = r\cos\theta_1 \sin\theta_2 \sin\theta_3 \\ x_2 = r\sin\theta_1 \sin\theta_2 \sin\theta_3 \\ x_3 = r\cos\theta_2 \sin\theta_3 \\ x_4 = r\cos\theta_3 \end{cases} \tag{7.35}$$

In this way it is possible to add more and more parameters, to express the relation in any number of dimensions, D (≥ 3). In analogy to (7.34), we have for particle j

$$E_{kp} = \sum_{i=1}^{D-1} \frac{r^2 A_{ji}^2 \dot{\theta_{ji}}^2}{2} \tag{7.36}$$

where the A_{ji}:s are constants, independent of r, $\dot{r}$ and the $\dot{\theta}_i$:s, but dependent of the θ_i:s. The kinetic energy due to the (collective) motion in radial direction is

$$E_{kr} = \frac{M\dot{r}^2}{2} \tag{7.37}$$

where M is the total mass of the particles. The potential energy of the particles is

$$E_p = M\frac{C_D}{2} r^\nu \tag{7.38}$$

where C_D is the constant that appears in (2.25). The constant "2" appears because of that the gravitational potential emerges from the particles themselfs. The Lagrangian then becomes

$$L = \frac{M\dot{r}^2}{2} + \sum_{j=1}^{N}\sum_{i=1}^{D-1} \frac{r^2 A_{ji}^2 \dot{\theta_{ji}}^2}{2} - M\frac{C_D}{2} r^\nu \tag{7.39}$$

where N is the number of particles. For the canonical momenta, we have

$$p_{ji} \equiv \frac{\partial L}{\partial \dot{\theta_{ji}}} = r^2 A_{ji}^2 \dot{\theta_{ji}} \tag{7.40}$$

and

$$p_r \equiv \frac{\partial L}{\partial \dot{r}} = M\dot{r} \tag{7.41}$$

We construct the Hamiltonian

$$H = \frac{p_r^2}{2M} + \sum_{j=1}^{N}\sum_{i=1}^{D-1} \frac{p_{ji}^2}{2A_{ji}^2 r^2} + M\frac{C_D}{2} r^\nu \tag{7.42}$$

According to (3.40) (with $m_j = r^2$, $n_{sr} = \text{const}$ (because the Hamiltonian formulation takes care of the spatial-volume dependence of r) and $n_m = N(D-1)$), the phase-space volume for the system, at energy E, is

$$g(E) \propto \int_a^{r_{max}} \mathrm{d}r\, (E - M\frac{C_D}{2}r^\nu)^{\frac{N(D-1)}{2}-\frac{1}{2}} r^{N(D-1)} \mathrm{d}E \tag{7.43}$$

where the "$\propto$" sign denotes that the two expressions differ by a multiplicative constant, independent of E and r. We have introduced an inner limiting sphere with radius a. We want to regard the cases where the energy, not a cutoff, sets an upper limit on r, that is, the cases where there may be negative specific heat. The kinetic energy, $(E - M\frac{C_D}{2}r^\nu)$, in (7.43) has to be positive for the complete interval of integration, from $r = a$ to the value on r for which the kinetic energy is zero, r_{max}. Since C_D has the same sign as ν, we can conclude that E has the same sign as ν. We can rewrite (7.43) as

$$\begin{aligned} g(E) &\propto \\ &\propto \int_a^{r_{max}} \mathrm{d}r\, (1 - \frac{C_D}{2}(\frac{r}{E^{\frac{1}{\nu}}})^\nu)^{\frac{N(D-1)}{2}-\frac{1}{2}} (\frac{r}{E^{\frac{1}{\nu}}})^{N(D-1)} E^{\frac{2+\nu}{2\nu}N(D-1)-\frac{1}{2}} \mathrm{d}E = \\ &= \mathrm{d}E\, E^{\frac{2+\nu}{2\nu}N(D-1)-\frac{1}{2}+\frac{1}{\nu}} \int_{a'}^{r'_{max}} \mathrm{d}r'\, (1 - \frac{C_D}{2}r'^\nu)^{\frac{N(D-1)}{2}-\frac{1}{2}} r'^{N(D-1)} \end{aligned} \tag{7.44}$$

where we have changed integration variable to $r' = rE^{-\frac{1}{\nu}}$. For r'_{max}, we have $1 - \frac{C_D}{2}r'^\nu_{max} = 0$, which implies that r'_{max} is a constant in E. When using a' $(= aE^{-\frac{1}{\nu}})$, the situation is more complicated.

We want to let $a \to 0$ in our calculations, while E shall be regarded as finite. To be able to do that, despite the fact that a together with other parameters defines the system and that we shall be able to vary E, we have to regard the definition of the system valid only for $|E| < A$, where A is some (eventually very big) constant. When we have set this A, we can define our system by chosing an a arbitralily close to zero, so that for any E fulfilling $|E| < A$, small changes in a do not significantly affect the specific heat, that is, we have the limit $a \to 0$ while E is finite.

The integrand in (7.44) goes strongly to zero when $\nu \geq -1$ and $r \to 0$. For $\nu = -2$ the integrand goes like r' when $r \to 0$. We then have

$$g(E) \propto \mathrm{d}E\, E^{\frac{2+\nu}{2\nu}N(D-1)+\frac{2-\nu}{2\nu}} \qquad \nu \geq -2 \tag{7.45}$$

With use of (3.31) and (3.32), we get

$$C_V = k(\frac{2+\nu}{2\nu}N(D-1) + \frac{2-\nu}{2\nu}) \qquad \nu \geq -2 \tag{7.46}$$

We see that the specific heat is negative for $\nu = -1$ for any D ($D \geq 3$). There is also a very small amount of negative specific heat, $C_V = -k$, for $\nu = -2$ for any D ($D \geq 3$). Gravitation obeys (2.28), and we have

$$C_V = k(\frac{4-D}{2(2-D)}N(D-1) + \frac{D}{2(2-D)}) \qquad \nu \geq -2 \tag{7.47}$$

For $\nu \leq -3$, the integrand can be approximated with $(-\frac{C_D}{2})^{\frac{N(D-1)}{2}-\frac{1}{2}} r'^{\frac{2+\nu}{2}N(D-1)-\frac{\nu}{2}}$ when $r \to 0$. This implies that

$$g(E) \propto \mathrm{d}E\, E^{\gamma}(D_1 + \frac{D_2}{\nu\gamma}(-\frac{C_D}{2})^{\frac{N(D-1)}{2}-\frac{1}{2}} a'^{\nu\gamma}) \qquad \nu \leq -3 \tag{7.48}$$

or

$$g(E) \propto \mathrm{d}E\, E^{\gamma}(D_1 + \frac{D_2}{\nu\gamma}(-\frac{C_D}{2})^{\frac{N(D-1)}{2}-\frac{1}{2}} a^{\nu\gamma} E^{-\gamma}) \qquad \nu \leq -3 \tag{7.49}$$

where $\gamma = \frac{2+\nu}{2\nu}N(D-1) + \frac{2-\nu}{2\nu}$, and D_1 and D_2 are constants in E, r, a and N. This implies that γ is positive and huge, and that D_1 and D_2 are finite. We want to use (3.41), which reads

$$C_V = k\frac{g'(E)^2}{g'(E)^2 - g(E)g''(E)} \tag{7.50}$$

From (7.49), we have

$$g'(E) \propto \mathrm{d}E\, D_1\gamma E^{\gamma-1} \tag{7.51}$$

and

$$g''(E) \propto \mathrm{d}E\, D_1\gamma(\gamma-1)E^{\gamma-2} \tag{7.52}$$

(7.50), (7.49), (7.51), and (7.52) give

$$C_V = k\frac{\gamma}{1 - \frac{D_2}{D_1}\frac{\gamma-1}{\nu\gamma}(-\frac{C_D}{2})^{\frac{N(D-1)}{2}-\frac{1}{2}} E^{-\gamma} a^{\nu\gamma}} \tag{7.53}$$

When $a \to 0$ this expression goes to zero. That is

$$C_V = 0 \qquad \nu \leq -3 \tag{7.54}$$

7.6 The Hertel/Thirring model

This model was first presented in Thirring [17]. In Hertel/Thirring [7], the model was generalised. Here, we present the former one. N particles with total energy E are confined inside a volume V. When two particles both are inside a certain part of V, V_0, they interact with each other, and feel a constant potential. This can be written

$$U(\vec{x}_i, \vec{x}_j) = -2C\theta_{V_0}(\vec{x}_i)\theta_{V_0}(\vec{x}_j) \tag{7.55}$$

where $U(\vec{x}_i, \vec{x}_j)$ is the potential energy due to interaction between particle i and j, C a constant, and the function θ_{V_0} is unity when the parameter is within V_0, otherwise zero. We denote the entropy of the system with S, and each particle's mass with m. We denote the number of particles in V_0 with N_0, the total energy of these particles E_0, and their entropy with S_0. We denote $V - V_0$ with V_1, $N - N_0$ with N_1, $E - E_0$ with E_1, and $S - S_0$ with S_1. We have

$$E_0 = \sum_{i=1}^{N_0} \frac{p_i^2}{2m} - CN_0^2 \tag{7.56}$$

and

$$E_1 = \sum_{i=N_0+1}^{N} \frac{p_i^2}{2m} \tag{7.57}$$

For the phase-space volumes, we have

$$g_0(E_0) = \frac{1}{N_0!} \int \mathrm{d}^{3N_0}x \, \mathrm{d}^{3N_0}p \, \delta(E_0 - \sum_{i=1}^{N_0} \frac{p_i^2}{2m} + CN_0^2) \tag{7.58}$$

and

$$g_1(E_1) = \frac{1}{N_1!} \int \mathrm{d}^{3N_1}x \, \mathrm{d}^{3N_1}p \, \delta(E_0 - \sum_{i=N_0+1}^{N} \frac{p_i^2}{2m}) \tag{7.59}$$

Integrating (7.58) and (7.59), and using (3.1), we get for the entropy of the two subsystems

$$S_0(E_0) = N_0 \ln V_0 - \frac{5}{2} N_0 \ln N_0 + \frac{3}{2} N_0 \ln(E_0 + CN_0^2) \tag{7.60}$$

and

$$S_1(E_1) = N_1 \ln V_1 - \frac{5}{2} N_1 \ln N_1 + \frac{3}{2} N_1 \ln E_1 \tag{7.61}$$

where we have omitted terms containing only m and N. The system is in the micro-canonical ensemble, and the entropy S is maximized subject to the constraints $N_0 + N_1 = N$ and $E_0 + E_1 = E$. If we denote $\frac{3T}{2NC} = t$ and $1 + \frac{E}{CN^2} = \epsilon$, we have

$$\begin{cases} t = \epsilon - 2\alpha + \alpha^2 \\ \epsilon = 2\alpha - \alpha^2 + \frac{3(1-\alpha)}{\ln(\frac{V_1}{V_0}\frac{1-\alpha}{\alpha})} \end{cases} \tag{7.62}$$

In Thirring [17], the following diagram is presented.

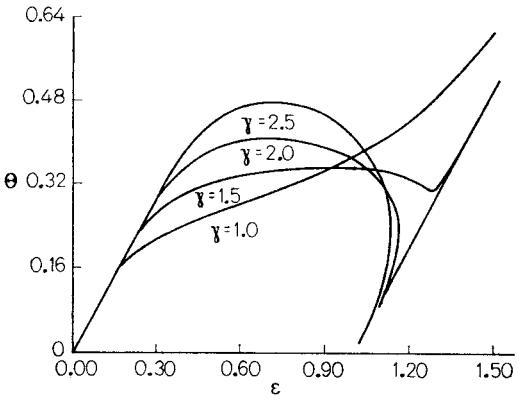

Fig. 6: This diagram is taken from Thirring [17]. Higher ϵ corresponds to higher energy, and θ is proportional to the temperature. The energy depends on N^γ. γ equals one corresponds to an extensive system. For higher γ, the energy is a superextensive (see chapter 6.1) parameter. In the model described above, γ equals two.

We see that this system has negative specific heat for those values on γ that make the energy become a non-extensive parameter.

7.7 The Aronson/Hansen model

In this model, first presented in Aronson/Hansen [2], we consider a system in the canonical ensemble consisting of N particles enclosed in a spherical shell with radius R. Each particle is a sphere with volume V_0 and mass m. The system is in internal equilibrium. That means, we have the same temperature, T, and local chemical potential, μ_{local}, (see below) for any subsystem. The density distribution, ρ, is then spherically symmetric, depending only on the radial coordinate, r. We regard subsystems in the grand canonical ensemble (fluctuating energy and particle number) that are so small that the gravitational potential does not vary over the subsystem, but so large that it contains very many particles. This model, we consider only in three dimensions, since it would involve a lot of complicated numerical computations to perform the corresponding calculations as the authors of [2] has performed for three-dimensional gravity. The gravitational potential energy of a particle is

$$E_p(r) = -\frac{mG}{r}\int_0^r \mathrm{d}r'\, 4\pi r'^2\rho(r') - mG\int_r^R \mathrm{d}r' 4\pi r'\rho(r') \tag{7.63}$$

The chemical potential, that is, the mean energy required to add a particle to a subsystem at location r, is

$$\mu = \mu_{local}(\rho(r), T) + E_p(r) = \text{constant} \tag{7.64}$$

where μ_{local} is the part of the chemical potential arising from kinetic energy and short-range forces between particles. μ is a constant, because we have equilibrium between subsystems. There is a thermodynamic relation that relates the pressure, P, and μ_{local}.

$$\frac{\partial P}{\partial \mu_{local}}|_{V,T} = \frac{\rho}{m} \tag{7.65}$$

where we hold the volume of the subsystem and the temperature constant. For convenience, we skip the bar sign in the rest of the calculation. (7.65) can be written

$$\frac{1}{\rho}\frac{\partial P}{\partial r} = \frac{1}{m}\frac{\partial \mu_{local}}{\partial r} \tag{7.66}$$

From (7.66), we can construct

$$\frac{1}{r^2}\frac{\partial}{\partial r}(\frac{r^2}{\rho}\frac{\partial P}{\partial r}) = \frac{1}{mr^2}\frac{\partial}{\partial r}(r^2\frac{\partial \mu_{local}}{\partial r}) \tag{7.67}$$

(7.67), (7.64) and (7.63) give

$$\frac{1}{r^2}\frac{\partial}{\partial r}(\frac{r^2}{\rho}\frac{\partial P}{\partial r}) = -4\pi G\rho \tag{7.68}$$

To write P in terms of ρ, we use a part of van der Waals equation of state, which resembles the equation of state of a perfect gas. In van der Waal's equation, repulsive forces between particles is represented by a correction of the volume of the system, so that the volume regarded as occupied by the particles is subtracted from the total volume. There is also a second correction, which we

will not use here. The part of van der Waal's equation of state that we are interested in, reads

$$PV(1 - \frac{\rho b}{m}) = N_0 kT \tag{7.69}$$

where $b = 4V_0$, V is the volume of the subsystem, N_0 the number of particles in the subsystem, and k Boltzmann's constant. (7.69) can be written

$$P(1 - \frac{\rho b}{m}) = \frac{\rho kT}{m} \tag{7.70}$$

Combining (7.68) and (7.70) give

$$\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}(\frac{b\rho}{m - b\rho} + \ln\frac{\rho}{m - b\rho}) = -4\pi G\beta\rho \tag{7.71}$$

With this non-linear differential equation, we can, by numerical computation, determine the density distribution, $\rho(r)$. The boundary conditions are

$$\frac{\partial\rho}{\partial r}|_{r=0} = 0 \tag{7.72}$$

and

$$-G\frac{Mm}{R^2} = -\frac{\partial E_p}{\partial r}|_{r=R} = \frac{\partial\mu_{local}}{\partial r}|_{r=R} = \frac{1}{\beta}\frac{\partial}{\partial r}(\frac{b\rho}{m - b\rho} + \ln\frac{\rho}{m - b\rho})|_{r=R} \tag{7.73}$$

where we have used a relation between force and field and some of the formulas above. (7.73) can also be regarded as an insurance that the total mass described by $\rho(r)$ is the same as the total mass, M $(= Nm)$. In Aronson/Hansen [2] the result of a numerical computation is presented, with $R = 60$ km, $N = 10^{57}$, m =neutron mass$= 1,67 \cdot 10^{-24}$ g and $b = 1,072 \cdot 10^{-39}$ cm^{-3}. The diagram in fig. 7 is taken from [2].

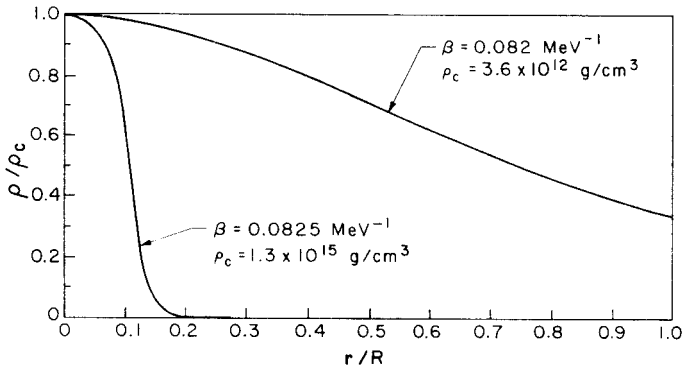

Fig. 7: This diagram is taken from Aronson/Hansen [2]. The density, ρ, in units of the central density, ρ_c, is plotted as a function of the distance from the centre, r, in units of the radius, R, of the outer limiting sphere. Two different temperatures are considered. The difference between the two curves represents a phase-transition.

We see a very dramatic change of the density distribution under a small change of the temperature, at a critical temperature. This is an illustration of a typical phenomena in self-gravitating systems in the canonical ensemble, namely a

phase-transition. This use to occur at a temperature where the same system in the micro- canonical ensemble has negative specific heat. The diagram in fig. 8, taken from [2], shows the relation between energy and temperature for the system.

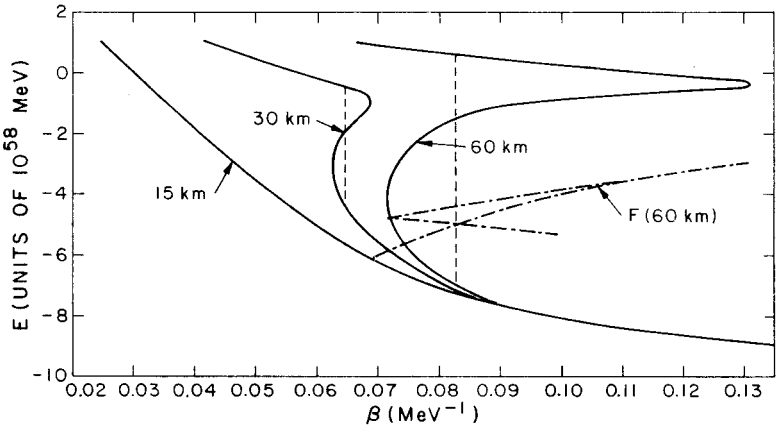

Fig. 8: This diagram is taken from Aronson/Hansen [2]. The energy, E, of the system is plotted as a function of the temperature parameter β $(= \frac{1}{kT})$ for three different radii of the outer limiting sphere.

If the energy is taken as the independent parameter (which is quite possible to do), we have a system in the micro-canonical ensemble, and as can be seen in the diagram, we have negative specific heat in some region of energy for the $R = 30$ km and $R = 60$ km case. If we take the temperature as the independent parameter, we have a system in the canonical ensemble (as was our original perspective on this system). Then, the energy as a function of the temperature is a multiple-valued function. The system follows a bransch of the function until the Helmholtz free energy (see chapter 3) is lower for another branch. Then the system jumps to the latter branch. This involves a huge positive value on the specific heat. As can be seen by (3.34), this is associated with large energy fluctuations, which can be interpreted as a necessary condition for the ability to jump in energy.

Chapter 8

Conclusions

This chapter mainly presents an article written on the subject "On the cause of negative specific heat in self-gravitating systems". (The title on the article may be another one.) The chapter is intended to contain the most important information for the reader to understand the conclusions, and information from other chapters is included here. If a little more comprehensive text is wanted, the reader may read previous chapters. If the reader has read the previous chapters, and understands the content, the reader may optionally read only the sections "Summary" and "Results" here. Fig. 9 represents a very good overview of the results achieved.

Summary

In the area of self-gravitating systems, the literature often gives the impression that long-range forces are the cause of negative specific heat in such systems. See, for instance, the very good review by Padmanabhan [15] and references given there. Here, we will show that this assumption is not quite satisfactory, since only some systems affected by long-range forces exhibit negative specific heat. Instead, we find that negative specific heat, for four models of systems with attracting particles, where the particles feel potentials on the form Cr^{ν}, exhibit negative specific heat only for $\nu = -1$.

It is, however, as often suggested, reasonable to believe that long-range forces in a system is the cause of non-extensivity of the system. With non-extensivity, we mean that if two originally separated subsystems with energy E_1 and E_2 are combined, their total energy will not be $E_1 + E_2$. For systems defined by short-range forces, the interaction energy between subsystems becomes negligible for large enough subsystems, since the interaction energy scales as the area of the subsystems, and then we have an extensive system. For subsystems interacting by long-range forces, the interaction energy remains significant, since the interaction energy scales as the volume of the subsystems, and we then have a non-extensive system.

Long- and short-range forces

Before investigating negative specific heat, we will define the concept of long-range and short-range forces. We limit our investigations to potentials on the form

$$\phi(r) = Cr^{\nu} \tag{8.1}$$

where r is the radial coordinate in a space with D dimensions ($D \geq 3$), ν an integer ($\nu \neq 0$), and C a constant (only depending on D and ν). For $D = 3$, $C = -GM$ and $\nu = -1$ we have the conventional non-relativistic gravitational potential from a point charge with mass M at $r = 0$. Let us regard a continuous medium with constant density, ρ, and investigate from which areas the potential energy of a particle embedded in the media at the point $r = 0$ comes. If only regarding contributions from spherical shells of radius Δ (and with the particle in the centre), and where $\epsilon \leq \Delta \leq R$, the potential energy of the particle with mass m is

$$U = -\int_{V_\Delta} \mathrm{d}V_\Delta \, \rho m C' r^{\nu} \propto \int_{\epsilon}^{R} \mathrm{d}r \, r^{D-1} r^{\nu} \propto R^{D+\nu} - \epsilon^{D+\nu} \qquad D + \nu \neq 0 \tag{8.2}$$

where we have omitted multiplicative constants (independent of r). Observe that (8.2) is valid for other forces than gravitation if "mass" is interpreted as the relevant charge in consideration, and ρ as the corresponding density. We assume $\epsilon \ll 1$ and $R \gg 1$, giving a strong dependence of ϵ for negative $D + \nu$, and a strong dependence of R for positive $D+\nu$. With this method, short-range forces are caracterised by

$$D + \nu < 0 \tag{8.3}$$

and long-range forces by

$$D + \nu > 0 \tag{8.4}$$

Results

We can use spaces with $D \geq 3$ and $\nu \neq 0$ as our theoretical "testbench" to investigate the cause of negative specific heat. There is, for instance, no reason why long-range forces should result in negative specific heat in three dimensions, but not in higher dimensions. If the relation is true only in three dimensions, there is probably another mechanism laying behind, that is more relevant to explain negative specific heat.

The definition of specific heat at constant volume is

$$C_V \equiv \frac{\partial < E >}{\partial T}|_V \tag{8.5}$$

where "<>" expresses a time average. We here use four models of self-gravitating systems, and we calculate in what regions of D and ν that the systems have negative specific heat. The models are "The Virial model", "The binary star model", "The Lynden-Bell model" and "The circular orbit model". The result is mapped in fig. 9. Long-range forces are defined by $D + \nu > 0$. Gravitation obeys $\nu = 2 - D$, and is therefore a long-range force. Only for $\nu = -1$ we have significant negative specific heat in our models.

As can be seen in fig. 9, there is a large region where long-range forces define the system, and where the system has positive specific heat. Therefore, it seems to be something more than the long-range nature of forces that is necessary for negative specific heat. For the Virial model and the Circular orbit model, the region of negative specific heat coincides with the region of negative total energy of the system. For the Binary star model and the Lynden-Bell model, it is possible to chose to study negative or positive energy of the system. We chose to study the case where there may be negative specific heat, that is, the case where no outer cutoff affects the system, and the energy has to be chosen negative for negative ν. In these cases, where all combinations of D and negative ν are studied with negative energy, negative specific heat is still present only for $\nu = -1$. In the Binary star model, The Lynden-Bell model and the Circular orbit model, the effective potential felt by a particle due to all the other particles, is on the form $\phi = Cr^{\nu}$, where r is the radial coordinate of the particle relative some fixed point, or centre of mass of the system. It would then be possible to suggest, that if the long-range nature of a force is sufficient to result in negative specific heat, this arises from emerging density distributions in the system that arise because of the long-range nature of the force, and which give an effective potential on another form than the one investigated here. These three models would then not be suitable for this kind of analyse. But, the Virial model, relying on the derivation of the Virial theorem for systems where the mutual interaction between pairs of particles is on the form Cr^{ν}, is valid independently of any emerging density distribution. This then becomes an argument that there has to be something more than the long-range nature of the force to give negative specific heat.

As mentioned before, long-range forces give rise to non-extensivity. In Thirring [17], a model with a potential on a quite another form than the one studied here is presented. In his model it is possible to change a parameter to vary the degree of non-extensivity of the energy of the system. Thirring's analyse shows that negative specific heat enters near to the point where the system becomes non-extensive.

The definition of long- and short-range forces used here is not certainly the relevant one. Since the systems studied are not homogeneous, other definitions not relying on a matter distribution with constant density may be more accurate.

All together, there is still no good explanation why the D and ν dependence of the sign of the specific heat is like our analysis shows it to be. Maybe there is a cause to negative specific heat that has more specifically to do with the parameter value $\nu = -1$.

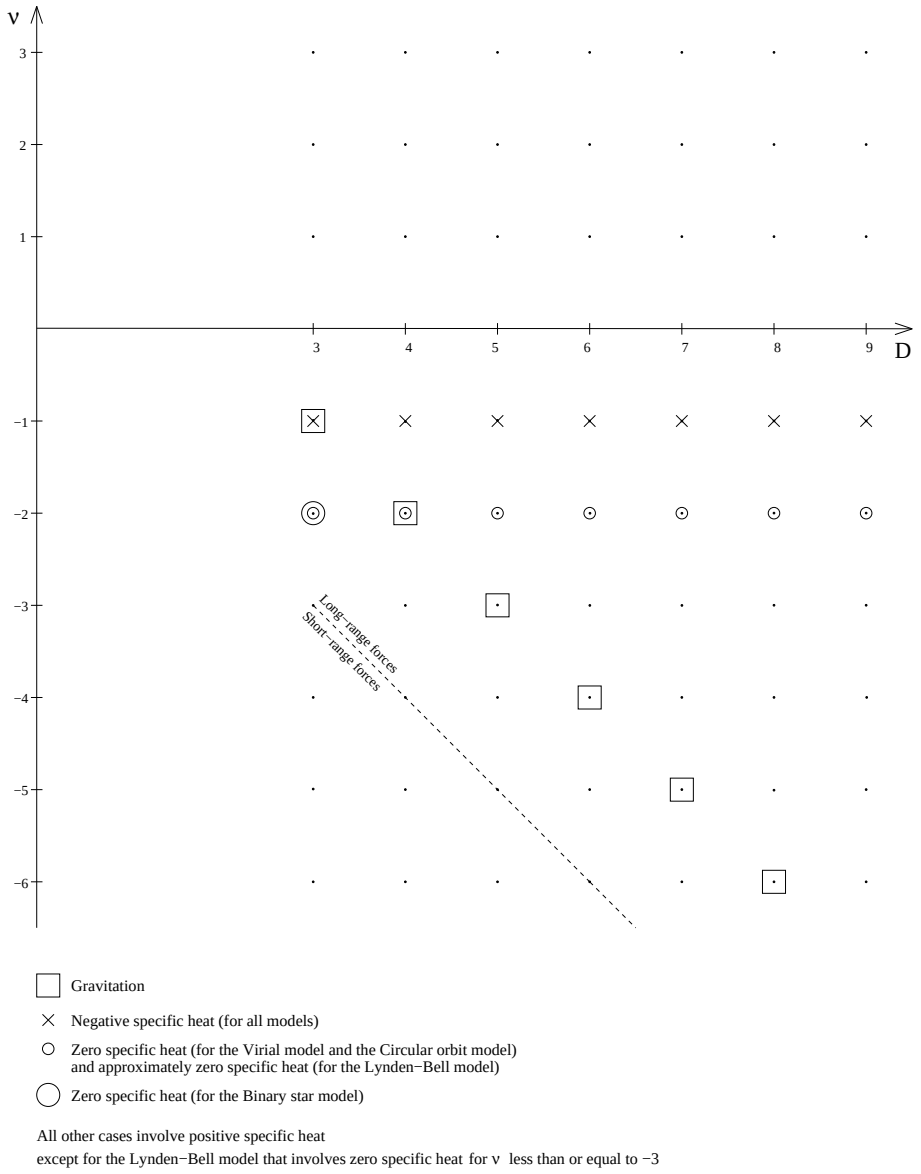

Fig. 9: Existence of negative specific heat as a function of D and ν for four models. The models are: The Virial model, The circular orbit model, The Lynden-Bell model and The Binary star model.

The Virial model

This model assumes a system of particles, where any pair of particles interact with a potential (8.1), where C and ν have the same sign. The Virial theorem applied to this system reads

$$< E >= \frac{\nu + 2}{\nu} < K > \tag{8.6}$$

In the micro-canonical ensemble, the energy, E, is negative for $\nu = -1$ for any D ($D \geq 3$). (8.5) and (8.6) give

$$C_V = \frac{\nu + 2}{\nu} \frac{\partial < K >}{\partial T}|_V \tag{8.7}$$

In the Virial model, we assume

$$\frac{\partial < K >}{\partial T}|_V > 0 \tag{8.8}$$

which is valid for many systems. We then conclude that C_V is negative exactly for $\nu = -1$. This coincides with negative total energy, E.

The binary star model

This model is presented for instance in Padmanabhan [15]. It contains two particles, each with mass m, that attract each other. In this essay, we generalise the model to D dimensions ($D \geq 3$) and a potential on the form (8.1), with arbitrary ν ($\nu \neq 0$), and where C and ν have the same sign. The Hamiltonian is

$$H(\vec{p_0}, \vec{q_0}, \vec{p_1}, \vec{q_1}) = \frac{p_0^2}{2m_0} + \frac{p_1^2}{2m_1} + mCr^\nu \tag{8.9}$$

where m_0 is the two particle's total mass ($= 2m$), m_1 their reduced mass ($= \frac{m}{2}$), $\vec{p_0}$ the conjugate linear momentum of their centre of mass, $\vec{p_1}$ their relative conjugate linear momentum, and r the distance between the two particles ($= q_1$). $\vec{q_0}$ is the position of the centre of mass. The particles are restricted to the volume of a sphere with much larger diameter than the maximum distance between the particles. The volume of the phase-space is

$$g^{(0)}(E) = C' \int_a^{r_{max}} \mathrm{d}r\, r^{D-1} (E - mCr^\nu)^{D-1} \tag{8.10}$$

We have in (8.10) introduced cutoffs, limits on r: an inner sphere with radius a, and an outer sphere with radius R. The outer sphere affects the system if $E > mCR^\nu$. Then $r_{max} = R$. If $E < mCR^\nu$, then $r_{max} = (\frac{E}{mC})^{\frac{1}{\nu}}$, which represents zero kinetic energy (the expression in parenthesis in (8.10) is zero for $r = r_{max}$). In this case, the energy of the system sets the upper limit on r.

To calculate the specific heat according to

$$C_V = k \frac{g^{(1)}(E)^2}{g^{(1)}(E)^2 - g^{(0)}(E) g^{(2)}(E)} \tag{8.11}$$

we need to know $g^{(1)}(E)(\equiv g'(E))$ and $g^{(2)}(E)(\equiv g''(E))$ as well as $g^{(0)}(E)(\equiv g(E))$.

$$g^{(1)}(E) = C'(D-1)\int_a^{r_{max}} \mathrm{d}r\, r^{D-1}(E - mCr^\nu)^{D-2} \tag{8.12}$$

$$g^{(2)}(E) = C'(D-1)(D-2)\int_a^{r_{max}} \mathrm{d}r\, r^{D-1}(E - mCr^\nu)^{D-3} \tag{8.13}$$

To solve the integrals, we use partial integration.

$$\begin{aligned}\int_a^{r_{max}} \mathrm{d}r\, r^{D-1}(E - mCr^\nu)^{n_j} &= \frac{1}{D}[r^D(E - mCr^\nu)^{n_j}]_a^{r_{max}} + \\ &+ \frac{n_j \nu m C}{D(D+\nu)}[r^{D+\nu}(E - mCr^\nu)^{n_j-1}]_a^{r_{max}} + \\ &+ \frac{n_j(n_j-1)\nu^2 m^2 C^2}{D(D+\nu)(D+2\nu)}[r^{D+2\nu}(E - mCr^\nu)^{n_j-2}]_a^{r_{max}} + \dots \\ &\dots + \frac{n_j(n_j-1)\cdot\ldots\cdot 1\cdot \nu^{n_j} m^{n_j} C^{n_j}}{D(D+\nu)(D+2\nu)\cdot\ldots\cdot(D+n_j\nu)}[r^{D+n_j\nu}]_a^{r_{max}}\end{aligned} \tag{8.14}$$

where n_j refers to the corresponding number in the formula for $g^{(j)}(E)$. This expression holds for cases where no denominator contains a zero factor, that is, when

$$D + i\nu \neq 0 \qquad i = 0, 1, \dots, n_j \tag{8.15}$$

We are interested in cases that may involve negative specific heat. This implies that the energy sets a limit on r, that is $r_{max} = (\frac{E}{mC})^{\frac{1}{\nu}}$. ($E$ has the same sign as ν, since C has the same sign as ν.) All terms in (7.24) that contains r_{max}, except for the last one, then becomes zero. Except for one r_{max} term, only terms containing powers of a remain. We let $a \to 0$. This can be regarded as representing an inner cutoff that really exists, but whose radius can be chosen arbitrarily close to zero.

We want to let $a \to 0$ in our calculations, while E shall be regarded as finite. To be able to do that, despite the fact that a together with other parameters defines the system and that we shall be able to vary E, we have to regard the definition of the system valid only for $|E| < A$, where A is some (eventually very big) constant. When we have set this A, we can define our system by chosing an a arbitralily close to zero, so that for any E fulfilling $|E| < A$, small changes in a do not significantly affect the specific heat, that is, we have the limit $a \to 0$ while E is finite.

It follows from the properties of the formula for the specific heat, (8.11), that it is possible to multiplicate $g^{(1)}(E)^2$ and $g^{(0)}(E)g^{(2)}(E)$ with one and the same factor without affecting C_V. More general, since $n_1 = D-2$, $n_0 = D-1$ and $n_2 = D-3$, multiplication of $g^{(j)}(E)$ with a factor $\gamma_1^{\gamma_2+\gamma_3 n_j}$ (γ_i arbitrary, but not dependent of j) do not change C_V. Since $a \to 0$, we can use this to simplify (8.14) by multiplying with a^{-l} where l is the lowest power of a in the terms in (8.14). Then the terms with a^l is transformed to terms constant in a, and all other terms goes to zero when $a \to 0$.

First, we consider $D + n_j\nu > 0$ for all j. We call this case A. Then the lowest power of a in (8.14) is 0. It appears in the last term, a term containing $r_{max}^{D+n_j\nu}$. The term in (8.14) that remain when $a \to 0$, is this term. We also multiply with

$\frac{D(D+\nu)\cdot\ldots\cdot(D+(D-3)\nu)}{C'(D-1)(D-2)\cdot\ldots\cdot 2(\nu mC)^{n_j}} r_{max}^{-(D+n_j\nu)}$. We then have

$$\begin{cases} g^{(0)} \sim \frac{1}{(D+(D-2)\nu)(D+(D-1)\nu)} \\ g^{(1)} \sim \frac{1}{D+(D-2)\nu} \\ g^{(2)} \sim 1 \end{cases} \tag{8.16}$$

where the "$\sim$" sign stands for equality after multiplication with an allowed factor and after taking the limit of a. For the specific heat, we have

$$C_V = k\frac{D+(D-1)\nu}{\nu} \qquad D+n_j\nu > 0 \tag{8.17}$$

This expression is positive for positive ν, and for negative ν negative in the complete interval where it is valid, namely for $\nu = -1$.

We then consider $D + n_j\nu < 0$ for all j. We call this case B. The terms that remain in (8.14) after multiplication with $a^{-(D+\nu n_j)}$ and when $a \to 0$ are the ones that contained $a^{D+\nu n_j}$, that is, one term from every integration bracket. We also multiply with $C'^{-1}(mC)^{-n_j}$. From (8.10), (8.12), (8.13) (8.14) and (8.11) we then have

$$\begin{cases} g^{(0)} \sim \frac{1}{D} + \frac{(D-1)\nu}{D(D+\nu)} + \frac{(D-1)(D-2)\nu^2}{D(D+\nu)(D+2\nu)} + \ldots + \frac{(D-1)(D-2)\cdot\ldots\cdot 1\cdot\nu^{D-1}}{D(D+\nu)\cdot\ldots\cdot(D+(D-1)\nu)} \\ g^{(1)} \sim (D-1)(\frac{1}{D} + \frac{(D-2)\nu}{D(D+\nu)} + \frac{(D-2)(D-3)\nu^2}{D(D+\nu)(D+2\nu)} + \ldots + \frac{(D-2)(D-3)\cdot\ldots\cdot 1\cdot\nu^{D-2}}{D(D+\nu)\cdot\ldots\cdot(D+(D-2)\nu)}) \\ g^{(2)} \sim (D-2)(D-1)(\frac{1}{D} + \frac{(D-3)\nu}{D(D+\nu)} + \frac{(D-3)(D-4)\nu^2}{D(D+\nu)(D+2\nu)} + \ldots \\ \qquad \ldots + \frac{(D-3)(D-4)\cdot\ldots\cdot 1\cdot\nu^{D-3}}{D(D+\nu)\cdot\ldots\cdot(D+(D-3)\nu)}) \end{cases} \tag{8.18}$$

The formula (8.14) is valid for a certain j only for cases where (8.15) is fulfilled for this j. The two cases that we calculated on above, demands that (8.15) is fulfilled for all j. Now, we will calculate on the case where there is one i $(1 \leq i \leq D-3)$ for which (8.15) is not fulfilled for any j We call this case C. Then, during the process of successive partial integrations, when the factor $D + (i-1)\nu$ appears in the denominator of a term, the next integration will result in a term on the form $C_1 \ln r$, where C_1 is a constant in r. When all the steps of integration have been performed, one term will be logarithmic, and this term will have the form $C_2 r^{\nu(n_j - i)} \ln r$, and will be the important one when we let $a \to 0$. When we have done all the allowed reductions of factors in the $g^{(j)}$:s, we obtain

$$\begin{cases} g^{(0)} \sim 1 \\ g^{(1)} \sim D-i-1 \\ g^{(2)} \sim (D-i-1)(D-i-2) \end{cases} \tag{8.19}$$

For the specific heat, we have

$$C_V = k(D-i-1) \qquad \exists i : D+i\nu = 0 \qquad 1 \leq i \leq D-3 \tag{8.20}$$

This expression is positive for all cases where it is valid.

Above, we have defined three categories to which the $g^{(j)}$:s can belong. Now, we investigate the cases for which all the $g^{(j)}$:s do not belong to the same category. We call this case D. When $a \to 0$, the most important terms in (7.24) are the ones with a logarithmic behavior in a, that means one term in every $g^{(j)}$ belonging to category C. The next most important kind of term is the one

with a polynomial dependence of a and with a negative exponent, that means terms in $g^{(j)}$:s belonging to category B. The least important kind of term is a one that is constant in a, namely the term remaining in $g^{(j)}$ when $a \to 0$ and $g^{(j)}$ belongs to category A. From this information we can conclude what $g^{(j)}$:s that dominates in (7.26) when calculating the specific heat. We then also use the following scheme, that is derived from (8.11):

$$\begin{cases} g^{(0)} \text{ most important } \Rightarrow C_V = 0 \\ g^{(1)} \text{ most important } \Rightarrow C_V = k \\ g^{(2)} \text{ most important } \Rightarrow C_V = 0 \\ g^{(0)} \text{ and } g^{(1)} \text{ most important } \Rightarrow C_V = k \\ g^{(0)} \text{ and } g^{(2)} \text{ most important } \Rightarrow C_V = 0 \\ g^{(1)} \text{ and } g^{(2)} \text{ most important } \Rightarrow C_V = k \\ g^{(0)} \text{ and } g^{(1)} \text{ and } g^{(2)} \text{ equally important } \Rightarrow \\ \qquad \Rightarrow C_V \text{ can be calculated according to cases A, B or C above} \end{cases} \tag{8.21}$$

In the table below, numerical values on C_V in terms of units of Boltzmann's constant, k, are presented. The type of case is also given.

ν	D							
	3	4	5	6	7	8	9	10
3	3.0 (A)	4.3 (A)	5.7 (A)	7.0 (A)	8.3 (A)	9.7 (A)	11 (A)	12 (A)
2	3.5 (A)	5.0 (A)	6.5 (A)	8.0 (A)	9.5 (A)	11 (A)	12 (A)	14 (A)
1	5.0 (A)	7.0 (A)	9.0 (A)	11 (A)	13 (A)	15 (A)	17 (A)	19 (A)
0	X	X	X	X	X	X	X	X
−1	−1 (A)	−1 (A)	−1 (A)	−1 (A)	−1 (A)	−1 (A)	−1 (A)	−1 (A)
−2	0 (D)	1 (D)	1 (D)	2 (C)	2.5 (B)	3 (C)	3.5 (B)	4 (C)
−3	1 (D)	1 (D)	2.3 (B)	3 (C)	3.7 (B)	4.3 (B)	5 (C)	5.7 (B)
−4	1 (D)	2 (C)	2.7 (B)	3.5 (B)	4.2 (B)	5 (C)	5.8 (B)	6.5 (B)
−5	1 (D)	2.3 (B)	3 (C)	3.8 (B)	4.6 (B)	5.4 (B)	6.2 (B)	7 (C)
−6	1 (D)	2.4 (B)	3.2 (B)	4 (C)	4.8 (B)	5.7 (B)	6.5 (B)	7.3 (B)

The Lynden-Bell model

This model was first presented in Lynden-Bell/Lynden-Bell [11]. N particles are, in three-dimensional space confined to a spherical surface of radius r. When comes to the gravitational interaction, the total mass of the particles, M, is assumed to be uniformly distributed over the surface. The radius, r, is fluctuating as a result of fluctuations in the distribution between potential and kinetic energy in the system.

Here, we generalise the model to a space with D ($D \geq 3$) dimensions, assuming the sphere to have one dimension less than the space has. We also generalise the potential to be on the form (8.1), where C and ν have the same sign. The Lagrangian of the system is

$$L = \frac{M\dot{r}^2}{2} + \sum_{j=1}^{N} \sum_{i=1}^{D-1} \frac{r^2 A_{ji}^2 \dot{\theta}_{ji}^2}{2} - M\frac{C}{2} r^\nu \tag{8.22}$$

where the $\dot{\theta}_{ij}$:s are angular velocities of particle j, the A_{ji}:s constants (independent of r, $\dot{r}$ and the $\dot{\theta}_{ji}$:s, but dependent of the θ_{ji}:s). The constant "2" in the

last term appears because of that the gravitational potential emerges from the particles themselfes. The Hamiltonian is

$$H = \frac{p_r^2}{2M} + \sum_{j=1}^{N} \sum_{i=1}^{D-1} \frac{p_{ji}^2}{2A_{ji}^2 r^2} + M\frac{C}{2} r^\nu \tag{8.23}$$

where we have introduced conjugate momentas. The phase-space volume is

$$g(E) \propto \int_a^{r_{max}} \mathrm{d}r \, (E - M\frac{C}{2} r^\nu)^{\frac{N(D-1)}{2} - \frac{1}{2}} r^{N(D-1)} \, \mathrm{d}E \tag{8.24}$$

where we have omitted constants (not dependent of r or E). We have introduced an inner limiting sphere with radius a. We want to regard the cases where the energy, not a cutoff, sets an upper limit on r, that is, the cases where there may be negative specific heat. The kinetic energy, $(E - M\frac{C}{2} r^\nu)$, in (8.24) has to be positive for the complete interval of integration, from $r = a$ to the value on r for which the kinetic energy is zero, r_{max}. Since C has the same sign as ν, we can conclude that E has the same sign as ν. We can rewrite (8.24) as

$$\begin{aligned} g(E) \propto & \\ & \int_a^{r_{max}} \mathrm{d}r \, (1 - \frac{C}{2} (\frac{r}{E^{\frac{1}{\nu}}})^\nu)^{\frac{N(D-1)}{2} - \frac{1}{2}} (\frac{r}{E^{\frac{1}{\nu}}})^{N(D-1)} E^{\frac{2+\nu}{2\nu} N(D-1) - \frac{1}{2}} \mathrm{d}E \; = \\ & = \mathrm{d}E \, E^{\frac{2+\nu}{2\nu} N(D-1) - \frac{1}{2} + \frac{1}{\nu}} \int_{a'}^{r'_{max}} \mathrm{d}r' \, (1 - \frac{C}{2} r'^\nu)^{\frac{N(D-1)}{2} - \frac{1}{2}} r'^{N(D-1)} \end{aligned} \tag{8.25}$$

where we have changed integration variable to $r' = rE^{-\frac{1}{\nu}}$. For r'_{max}, we have $1 - \frac{C}{2} r'^\nu_{max} = 0$, which implies that r'_{max} is a constant in E. When using a' $(= aE^{-\frac{1}{\nu}})$, the situation is more complicated.

We want to let $a \to 0$ in our calculations, while E shall be regarded as finite. To be able to do that, despite the fact that a together with other parameters defines the system and that we shall be able to vary E, we have to regard the definition of the system valid only for $|E| < A$, where A is some (eventually very big) constant. When we have set this A, we can define our system by chosing an a arbitralily close to zero, so that for any E fulfilling $|E| < A$, small changes in a do not significantly affect the specific heat, that is, we have the limit $a \to 0$ while E is finite.

The integrand in (8.25) goes strongly to zero when $\nu \geq -1$ and $r \to 0$. For $\nu = -2$ the integrand goes like r' when $r \to 0$. We then have

$$g(E) \propto \mathrm{d}E \, E^{\frac{2+\nu}{2\nu} N(D-1) + \frac{2-\nu}{2\nu}} \qquad \nu \geq -2 \tag{8.26}$$

With use of the definition of temperature and (8.5), we get

$$C_V = k(\frac{2+\nu}{2\nu} N(D-1) + \frac{2-\nu}{2\nu}) \qquad \nu \geq -2 \tag{8.27}$$

We see that the specific heat is negative for $\nu = -1$ for any D $(D \geq 3)$. There is also a very small amount of negative specific heat, $C_V = -k$, for $\nu = -2$ for any D $(D \geq 3)$.

For $\nu \leq -3$, the integrand in (8.25) can be approximated with $(-\frac{C}{2})^{\frac{N(D-1)}{2}-\frac{1}{2}} r'^{\frac{2+\nu}{2}N(D-1)-\frac{\nu}{2}}$ when $r \to 0$. This implies that

$$g(E) \propto \mathrm{d}E\, E^{\gamma}(D_1 + \frac{D_2}{\nu\gamma}(-\frac{C}{2})^{\frac{N(D-1)}{2}-\frac{1}{2}} a^{\nu\gamma} E^{-\gamma}) \qquad \nu \leq -3 \tag{8.28}$$

where $\gamma = \frac{2+\nu}{2\nu}N(D-1) + \frac{2-\nu}{2\nu}$, and D_1 and D_2 are constants in E, r, a and N. This implies that γ is positive and huge, and that D_1 and D_2 are finite. We want to use (8.11). From (8.28), we have

$$g'(E) \propto \mathrm{d}E\, D_1 \gamma E^{\gamma-1} \tag{8.29}$$

and

$$g''(E) \propto \mathrm{d}E\, D_1 \gamma(\gamma-1) E^{\gamma-2} \tag{8.30}$$

(8.11), (8.28), (8.29), and (8.30) give

$$C_V = k\frac{\gamma}{1 - \frac{D_2}{D_1}\frac{\gamma-1}{\nu\gamma}(-\frac{C}{2})^{\frac{N(D-1)}{2}-\frac{1}{2}} E^{-\gamma} a^{\nu\gamma}} \tag{8.31}$$

When $a \to 0$ this expression goes to zero. That is

$$C_V = 0 \qquad \nu \leq -3 \tag{8.32}$$

The circular orbit model

In this model, we assume one particle in a circular orbit in the potential (8.1), where C and ν have the same sign. This model lacks a physical mechanism that makes the system ergodic. The ergodicity has to be assumed explicitly. We assume that the particle is confined spatially within a distance r to $r + \mathrm{d}r'$ from the centre, which gives for the spatial volume of the system

$$V_r \propto r^{D-1}\mathrm{d}r' \tag{8.33}$$

where we have omitted multiplicative constants (independent of r, $\mathrm{d}r'$ and the total energy of the particle, E). We assume the length of the linear momentum of the particle to be in the interval $|\vec{p}|$ to $|\vec{p}| + \mathrm{d}|\vec{p}|$. $\vec{p}$ is directed perpendicular to the central force potential's radius, so that the particle moves at the surface of a sphere in a D-dimensional space. This surface has $D-1$ dimensions. Except for a length in an interval of size $\mathrm{d}|\vec{p}|$, the direction of $\vec{p}$ is arbitrary as long as it is a tangent to the $(D-1)$-dimensional surface. The change of direction of $\vec{p}$ is then performed in $D-2$ dimensions, and $|\vec{p}|$ is the radius defining the size of this $(D-2)$-dimensional space. For the size of the momentum space, V_p, we then have

$$V_p \propto |\vec{p}|^{D-2}\frac{\mathrm{d}|\vec{p}|}{\mathrm{d}r}\mathrm{d}r' \tag{8.34}$$

Provided that the particle is in V_r and in V_p and $\mathrm{d}r'$ is infinitesimal, the spatial coordinates and the momentum coordinates are uncorrelated, which can be expressed

$$V_{rp} = V_r V_p \tag{8.35}$$

where V_rp is the volume of the system's phase-space. Using some common relations from classical mechanics, we get

$$|\vec{p}| \propto r^{\frac{\nu}{2}} \tag{8.36}$$

From (8.36), we get

$$\frac{\mathrm{d}|\vec{p}|}{\mathrm{d}r} \propto r^{\frac{\nu}{2}-1} \tag{8.37}$$

(8.33), (8.34), (8.35), (8.36) and (8.37) give

$$V_{rp} \propto r^{D-1+\frac{\nu(D-2)}{2}+\frac{\nu}{2}-1}\mathrm{d}r'^2 \tag{8.38}$$

We need to relate $\mathrm{d}r'$ to r. The phase-space volume of a system, $g(E)$, is dimensionless. To obtain this, for any system, expressions on for instance the form $\frac{C'Mm}{a^{-\nu}E}$ occurs in the expression for $g(E)$, where C' is the gravitational constant, M and m masses, a a distance, and E the total energy of the system. The relevant aspect of this is here that when distances appear in models of self-gravitating systems, they are multiplied by the energy of the system. This means, that if we scale the energy of the system, we can keep the properties of the system by the inverse scaling of the spatial coordinates in the model. (This resembles "mechanical similarity", see Landau/Lifshitz [9].) It is then reasonable to think that $\mathrm{d}r'$ has to be proportional to r. So, we have

$$\mathrm{d}r' \propto r \tag{8.39}$$

By use of (8.39), we can rewrite (8.38) as

$$V_{rp} \propto r^{D-1+\frac{\nu(D-2)}{2}+\frac{\nu}{2}}\mathrm{d}r \tag{8.40}$$

There is a relation between r and the total energy.

$$E \propto r^{\nu} \tag{8.41}$$

(8.40) and (8.41) give the temperature.

$$T = \frac{E}{\frac{D-1}{\nu}+\frac{D}{2}-\frac{1}{2}} \tag{8.42}$$

The specific heat is

$$C_V = \frac{D-1}{\nu}+\frac{D}{2}-\frac{1}{2} \tag{8.43}$$

We see that the specific heat is negative for $\nu = -1$ for any D ($D \geq 3$). Using (8.42) and (8.43) give

$$T = \frac{E}{C_V} \tag{8.44}$$

Since the temperature of a self-gravitating system is positive, we can conclude that this system always has the same sign on E and C_V.

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