Electrodynamics HT22
Assignment 3

1 Contravariance, Covariance

Consider arbitrary transformations of space or spacetime, i.e.

$$x^\alpha \to x'^\alpha = x'^\alpha(x). \tag{1}$$

Define contravariant vectors as objects that transform according to

$$V'^\alpha(x') = \frac{\partial x'^\alpha}{\partial x^\beta} V^\beta(x). \tag{2}$$

1. Show that in the special case of Lorentz transformations we recover the transformation of a Lorentz vector.
2. Show how covariant vectors must transform in order to guarantee that $V^\alpha U_\alpha$ transforms like a scalar.
3. Show that Kronecker's delta δ^α_β is an invariant tensor under the general transformation.

Solution

1. To show that in the special case of Lorentz transformations we recover the transformation of a Lorentz vector we simply use the Lorentz transformation $x' = \Lambda x$ in the definition of contravariant vectors Equation 2. We compute

$$\begin{aligned} V'^\alpha(x') &= V'^\alpha(\Lambda x) = \frac{\partial x'^\alpha}{\partial x^\beta} V^\beta(x) \\ &= \frac{\partial}{\partial x^\beta}\left(\Lambda^\alpha_\gamma x^\gamma\right) V^\beta(x) = \Lambda^\alpha_\gamma \frac{\partial x^\gamma}{\partial x^\beta} V^\beta(x) \\ &= \Lambda^\alpha_\gamma \delta^\gamma_\beta V^\beta(x) = \Lambda^\alpha_\beta V^\beta(x) \end{aligned} \tag{3}$$

and find that the resulting expression is a Lorentz transformation.

2. To find out how covariant vectors must transform in order to guarantee that $V^\alpha U_\alpha$ transforms like a scalar we make a general transformation of V^α and U_α. The vector V^α is contravariant and transforms according to Equation 2, and U_α we allow for a linear transformation with a new tensor called $\tilde{\Lambda}$

$$\begin{aligned} &V'^\alpha(x') = \Lambda^\alpha_\beta V^\beta(x) \\ &U'_\alpha(x') = \tilde{\Lambda}^\gamma_\alpha U_\gamma(x) \\ &\to V'^\alpha(x')\, U'_\alpha(x') = \Lambda^\alpha_\beta V^\beta(x) \tilde{\Lambda}^\gamma_\alpha U_\gamma(x) \end{aligned} \tag{4}$$

For $V^\alpha U_\alpha$ to transform like a scalar

$$\Lambda^\alpha_\beta V^\beta(x) \tilde{\Lambda}^\gamma_\alpha U_\gamma(x) = V^\beta(x) U_\beta(x) \tag{5}$$

must hold. This implies

$$\begin{aligned} &\Lambda^\alpha_\beta \tilde{\Lambda}^\gamma_\alpha = \mathbb{I}^\gamma_\beta \\ &\to \Lambda\tilde{\Lambda} = \mathbb{I} \Rightarrow \tilde{\Lambda} = \Lambda^{-1}. \end{aligned} \tag{6}$$

Therefore covariant vectors must transform as

$$U'_\alpha(x') = \left(\Lambda^{-1}\right)^\beta_\alpha U_\beta(x). \tag{7}$$

3. To show that Kronecker's delta δ^α_β is an invariant tensor under the general transformation we make a general tensor transformation of it. The definition of δ^α_β is that it is zero for $\alpha \neq \beta$ and one for $\alpha = \beta$. We find

$$\delta'^\alpha_\beta = \Lambda^\alpha_\mu \left(\Lambda^{-1}\right)^\nu_\beta \delta^\mu_\nu = \Lambda^\alpha_\mu \left(\Lambda^{-1}\right)^\mu_\beta = \delta^\alpha_\beta, \tag{8}$$

which is what we expected. Kronecker's delta is invariant under the transformation.

2 Transforming Scalars

Consider a scalar field having the form

$$\phi(x) = \frac{1}{r^2}, \quad r^2 = x^2 + y^2 + z^2. \tag{9}$$

Perform a Lorentz boost in the $t-x$-plane, and express the new function ϕ' that you obtain in this way as a function of the coordinates (t, x, y, z). What does the new function look like?

Solution

Take a Lorentz boost in the x-direction of frame S. The new frame S' moves with velocity v in the x-direction of frame S. We have the coordinate transformation[1], where we set $c = 1$,

$$\begin{aligned} t' &= \gamma(t + vx) \\ x' &= \gamma(x + vt) \\ y' &= y \\ z' &= z, \end{aligned} \tag{10}$$

and its inverse

$$\begin{aligned} t &= \gamma\left(t' - vx'\right) \\ x &= \gamma\left(x' - vt'\right) \\ y &= y' \\ z &= z'. \end{aligned} \tag{11}$$

In our case we have $\phi(x) = 1/r^2$ and performing the Lorentz boost we find

$$\phi'\left(x'\right) = \phi(x) = \frac{1}{r^2} = \frac{1}{\gamma^2\left(x' - vt'\right)^2 + y'^2 + z'^2} \tag{12}$$

Here we can rename the primed variables, $x' \to x$, and then we have $\phi'(x)$

$$\phi'(x) = \frac{1}{\gamma^2(x - vt)^2 + y^2 + z^2}. \tag{13}$$

Visualisation

If we want to see what our first function $\phi(x)$ looks like we can first make it two-dimensional, by setting $z = 0$ and then making a contour plot of $f(x, y) = 1/\left(x^2 + y^2\right)$. What we find are circles centered on the origin, see Figure 1. Similarly, in three dimensions the contours are spheres (but we cannot plot this since it would require a four-dimensional space).

[1] The expressions for the boost (and its inverse) are also valid if we change $v \to -v$, and interchange the primed and unprimed variables. This is the case because the transformation must hold between any two inertial frames for any v from $-c$ to c.

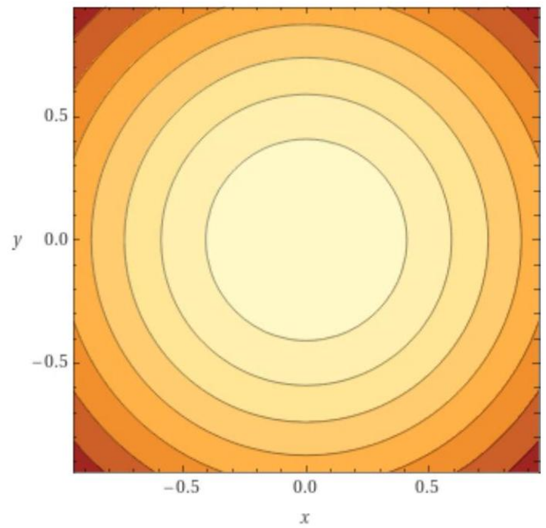

Figure 1: Contour plot of $f(x, y) = 1/\left(x^2 + y^2\right)$. Here $z = 0$.

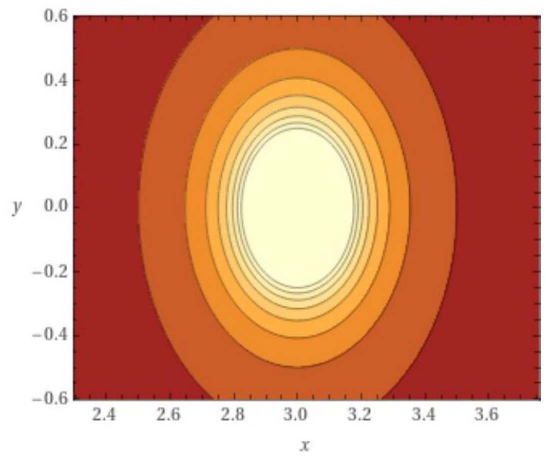

Figure 2: Contour plot of $f(x, y) = 1/\left(2(x-3)^2 + y^2\right)$. Here $z = 0, vt = 3$ and $\gamma = 2$.

The boosted function $\phi'(x)$ looks a bit different. We can see from its form that it is centered on $(vt, 0, 0)$ in the (x, y, z)-space. Setting $z = 0$ and making a contour plot we find that the contours of $\phi'(x)$ are ellipsoids contracted in the boosted direction, here it is the x-direction. For a twodimensional example, see Figure 2. γ is the factor that contracts the ellipsoid in the x-direction. In the limit $v \to c$ we have $\gamma \to \infty$ and the ellipsoids get squashed to circular discs in the (y, z)-plane at $x = vt$.

3 Transforming Fields

We are given electric and magnetic fields, and are asked to perform a Lorentz boost in the $t-x$-plane. The resulting transformed fields should then be expressed as functions of the coordinates (t, x, y, z).

1. The electromagnetic field from a point charge at rest at the origin,

$$\vec{E}(x) = \frac{1}{r^3}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{\hat{e}_r}{r^2}, \qquad \vec{B}(x) = \vec{0}. \tag{14}$$

2. The electromagnetic field

$$\vec{E} = \cos(t - x)\hat{e}_2 \qquad \vec{B} = \cos(t - x)\hat{e}_3, \tag{15}$$

which is a wave propagating in x-direction.

Solution

Before discussing the two specific field configurations, let us recap what an x-boost does to the electromagnetic fields. Setting $c = 1$, the Lorentz boost's matrix representation is

$$\Lambda^\mu_\nu = \begin{pmatrix} \gamma & \gamma v & 0 & 0 \\ \gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{16}$$

for velocity v and $\gamma = (1 - v^2)^{-\frac{1}{2}}$. This corresponds to the coordinate transformation

$$\begin{aligned} t' &= \gamma(t + vx) \\ x' &= \gamma(x + vt) \\ y' &= y \\ z' &= z. \end{aligned} \tag{17}$$

We cannot transform the fields directly, since $\vec{E}$ and $\vec{B}$ are not really independent. Instead we need to transform the electromagnetic tensor,

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}^{\mu\nu}, \tag{18}$$

following the standard rules for tensor transformations,

$$F'^{\mu\nu}(x') = \Lambda^\mu_\alpha \Lambda^\nu_\beta F^{\alpha\beta}(x). \tag{19}$$

This is equivalent to the matrix multiplication $F'(x') = \Lambda F(x) \Lambda^T$, which gives

$$\begin{aligned} F'(x') &= \Lambda F(x) \Lambda^T && (20a)\\ &= \begin{pmatrix} \gamma & \gamma v & 0 & 0 \\ \gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} \begin{pmatrix} \gamma & \gamma v & 0 & 0 \\ \gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} && (20b)\\ &= \begin{pmatrix} 0 & E_x & \gamma\left(E_y + vB_z\right) & \gamma\left(E_z - vB_y\right) \\ -E(x) & 0 & \gamma\left(B_z + vE_y\right) & \gamma\left(-B_y + vE_z\right) \\ -\gamma\left(E_y + vB_z\right) & -\gamma\left(vE_y + B_z\right) & 0 & B_x \\ -\gamma\left(E_z - vB_y\right) & -\gamma\left(vE_z - B_y\right) & -B_x & 0 \end{pmatrix}. && (20c) \end{aligned}$$

We can now read off the new field configurations,

$$\vec{E}'(\vec{x}') = \begin{pmatrix} E_x(x) \\ \gamma\left(E_y(x) + vB_z(x)\right) \\ \gamma\left(E_z(x) - vB_y(x)\right) \end{pmatrix}, \quad \text{and} \quad \vec{B}'(\vec{x}') = \begin{pmatrix} B_x(x) \\ \gamma\left(B_y(x) - vE_z(x)\right) \\ \gamma\left(B_z(x) + vE_y(x)\right) \end{pmatrix}. \tag{21}$$

Note that we still need to fix the coordinate dependence, since the primed fields need to be expressed in the primed coordinates. Thus, we invert the coordinate transformation given in Equation 17, and have to plug in

$$\begin{aligned} t &= \gamma(t' - vx') \\ x &= \gamma\left(x' - vt'\right) \\ y &= y' \\ z &= z'. \end{aligned} \tag{22}$$

Let us turn to the specific fields we are given.

1. The tensor transformation gives

$$\begin{aligned} E'_x\left(x'\right) &= E_x(x) = \frac{x}{r^3} \\ E'_y\left(x'\right) &= \gamma\left(E_y(x) + vB_z(x)\right) = \frac{\gamma y}{r^3} \\ E'_z\left(x'\right) &= \gamma\left(E_z(x) - vB_y(x)\right) = \frac{\gamma z}{r^3} \end{aligned} \tag{23}$$

   and

$$\begin{aligned} B'_x\left(x'\right) &= B_x(x) = 0 \\ B'_y\left(x'\right) &= \gamma\left(B_y(x) - vE_z(x)\right) = -\frac{\gamma v z}{r^3} \\ B'_z\left(x'\right) &= \gamma\left(B_z(x) + vE_y(x)\right) = \frac{\gamma v y}{r^3} \end{aligned} \tag{24}$$

   Changing the coordinate dependence to x' etc., we find

$$\vec{E}'(x') = \frac{\gamma}{\sqrt{\gamma^2\left(x' - vt'\right)^2 + y'^2 + z'^2}^3} \begin{pmatrix} x' - vt' \\ y' \\ z' \end{pmatrix} \tag{25}$$

   and

$$\vec{B}'(x') = \frac{\gamma v}{\sqrt{\gamma^2\left(x' - vt'\right)^2 + y'^2 + z'^2}^3} \begin{pmatrix} 0 \\ -z' \\ y' \end{pmatrix}. \tag{26}$$

   In Figure 3 we have the field $E(x)$ and $E'(x')$ with $\gamma = 3$ and $vt = 0$. From the figures we can observe that the field lines from the boosted point charge are more densely packed in the plane orthogonal to the direction of the boost, here x-direction. A moving charged particle constitutes a current, and indeed we see an accompanying magnetic field around the boosted particle in Figure 4.

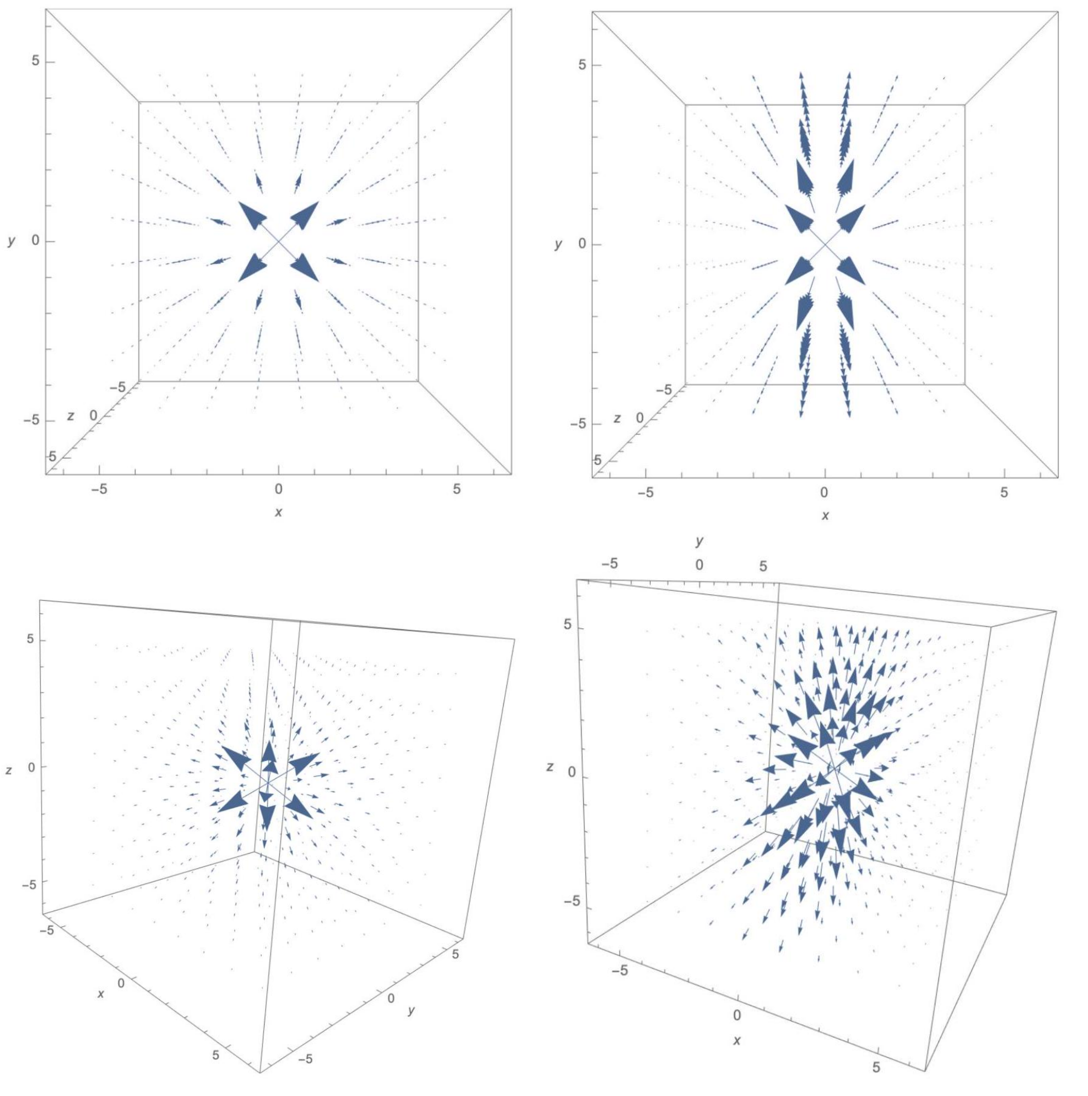

Figure 3: 3D plots of the electric fields, Left: $E(x)$, viewed from the xy-plane (top), and at some angle (bottom). Right: $E'(x')$, viewed from the $x'y'$-plane (top), and at an angle (bottom). Here $\gamma = 3$ and $vt = 0$. The particle is moving in the x'-direction with velocity v.

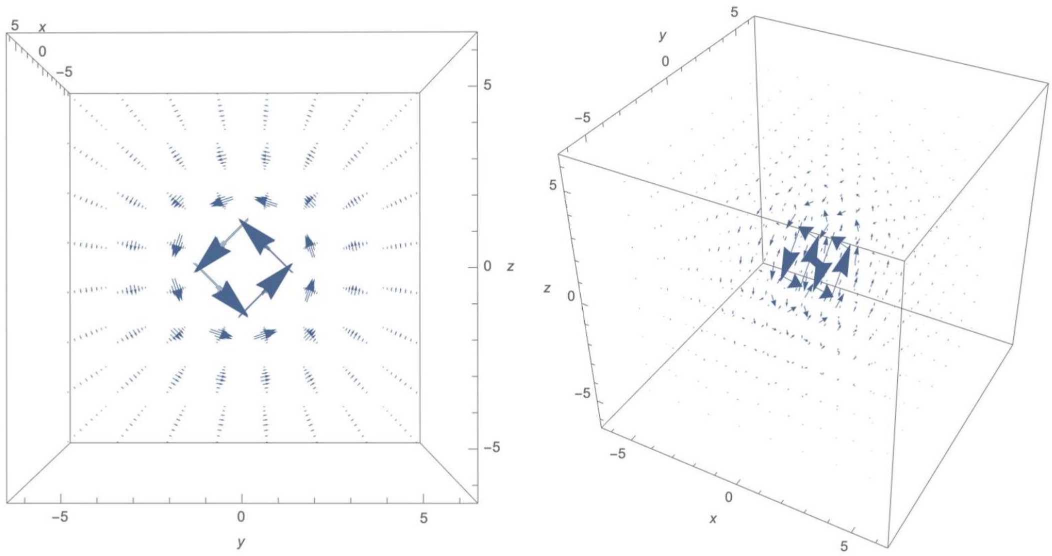

Figure 4: 3D plot of the vector field $B'(x)$, viewed from the $y'z'$-plane(top) and an angle (bottom). Here $v = 0.6(\gamma = 1.25)$. The particle is moving in the x-direction with velocity v.

2. We repeat the calculation for the propagating wave,

$$\vec{E}(x) = \cos(t - x)\hat{e}_y \quad \text{and} \quad \vec{B}(x) = \cos(t - x)\hat{e}_z. \tag{27}$$

The tensor transformation leads us to

$$\vec{E}'(x') = (1 + v)\gamma\cos(t - x)\hat{e}_y \quad \text{and} \quad \vec{B}'(x') = (1 + v)\gamma\cos(t - x)\hat{e}_z. \tag{28}$$

We change coordinates to x' using the inverse boost $x(x')$ in Equation 22 again, which gives $t - x = \gamma(1 + v)\,(t' - x')$. Note that

$$\gamma(1 + v) = \sqrt{\frac{1 + v}{1 - v}}. \tag{29}$$

The final fields are

$$\begin{aligned} \vec{E}'(x') &= \sqrt{\frac{1 + v}{1 - v}}\cos\left(\sqrt{\frac{1 + v}{1 - v}}(t - x)\right)\hat{e}_y \\ \vec{B}'(x') &= \sqrt{\frac{1 + v}{1 - v}}\cos\left(\sqrt{\frac{1 + v}{1 - v}}(t - x)\right)\hat{e}_z \end{aligned} \tag{30}$$

Recall that $|v| \leq c = 1$. The factor $\sqrt{\frac{1+v}{1-v}}$ is called the Doppler factor. In our equations for $\vec{E}'(x)$ and $\vec{B}'(x)$ this factor increases ($v > 0$) or decreases ($v < 0$) the amplitude and frequency of the fields, compared to $E(x)$ and $B(x)$. For $v > 0$ the wave is therefore blueshifted, and for $v < 0$ it is redshifted. Here $v < 0$ means that the sources is moving away from the observer and $v > 0$ that the source is moving towards the observer.

Observing Waves

We can see the difference between $E(x)$ and $E'(x)$ from a stationary observer's perspective. In Figure 5 (a) we can see how a static observer sees $E(x)$ during proper time τ. When we make a Lorentz transformation of a point the point is transported along a hyperbola, see Figure 5 (b). Therefore, in Figure 5 (c) we see the transformed wave $E'(x)$ pass a stationary observer, during proper time τ'. The transformed wave passes by the observer faster since $\tau' < \tau$ in the figure. If we were to transform both the wave and observer we would just end up with the same proper time τ for observation, i.e. we would still have Figure 5 (a).

The wave fields

$$\vec{E} = f(t-x)\hat{e}_y, \quad \vec{B} = f(t-x)\hat{e}_z \tag{31}$$

always constitue a solution to Maxwell's equations, for arbitrary function f. This is a plane electromagnetic wave propagating in the x-direction. We can show that Maxwell's equations are fulfilled ($c = 1$)

$$\begin{aligned}
&\bar{\nabla} \cdot \bar{B} = \partial_z f(t-x) = 0 \\
&\bar{\nabla} \times \bar{E} + \partial_t \bar{B} = \partial_x E_y - \partial_y E_x + B_z = -f' + f' = 0 \\
&\bar{\nabla} \cdot \bar{E} = \partial_y f(t-x) = 0 \\
&\bar{\nabla} \times \bar{B} - \partial_t \bar{E} = \partial_z B_x - \partial_x B_z - \partial_t E_y = f' - f' = 0.
\end{aligned} \tag{32}$$

It is also clear that $\bar{E} \cdot \bar{B} = 0$. We can easily calculate the energy density of the wave

$$U = \frac{1}{8\pi}\left(\bar{E}^2 + \bar{B}^2\right) = \frac{1}{4\pi} f^2 \tag{33}$$

and the Poynting vector

$$\bar{S} = \frac{1}{4\pi} \bar{E} \times \bar{B} = \frac{1}{4\pi} \begin{pmatrix} f^2 \\ 0 \\ 0 \end{pmatrix} \tag{34}$$

which points in the direction of the boost, here the x-direction.

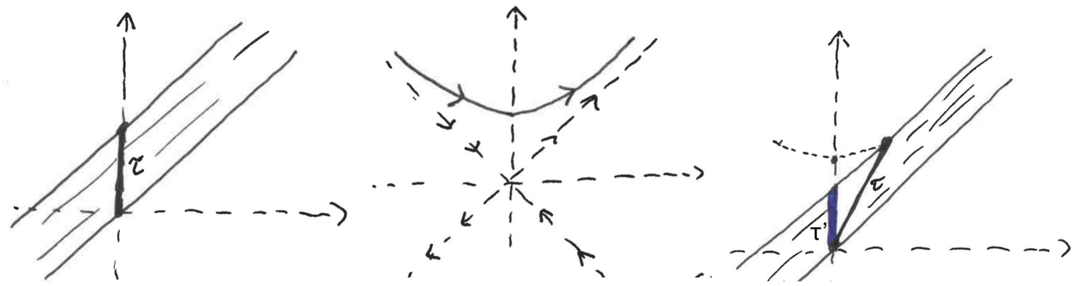

Figure 5: (a) Stationary observer observes the wave $E(x)$ during proper time τ
(b) Lorentz transformed points move along hyperbolae. (c) A stationary observer observes the transformed wave $E'(x)$ during proper time τ'. Notice that $\tau' < \tau$.

Acknowledgement

The problems are taken from Ingemar's notes and Jackson's book.
Solutions are adapted from a previous tutor, Nadia Flodgren.