Electrodynamics HT22
Assignment 2

1 Jackson, Problem 5.1

Starting with the differential expression

$$\mathrm{d}\mathbf{B} = \underbrace{\frac{\mu_0 I}{4\pi}}_{=:k} \mathrm{d}\mathbf{l}' \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} \tag{1}$$

for the magnetic induction at the point P with coordinate $\mathbf{x}$ produced by an increment of current $I\mathrm{d}\mathbf{l}'$ at $\mathbf{x}'$, show explicitly that for a closed loop carrying a current I the magnetic induction at P is

$$\mathbf{B} = \frac{\mu_0 I}{4\pi}\nabla\Omega \tag{2}$$

where Ω is the solid angle subtended by the loop at the point P. This corresponds to a magnetic scalar potential, $\Phi_M = -\mu_0 I\Omega/4\pi$. The sign convention for the solid angle is that Ω is positive if the point P views the "inner" side of the surface spanning the loop, that is, if a unit normal $\mathbf{n}$ to the surface is defined by the direction of current flow via the right-hand rule, Ω is positive if $\mathbf{n}$ points away from the point P, and negative otherwise. This is the same convention as in Section 1.6 (Jackson) for the electric dipole layer.

Solution

Let us write the differential equation in terms of vector components

$$dB_i = \hat{e}_i \cdot \mathrm{d}\mathbf{B} = \frac{\mu_0 I}{4\pi}\hat{e}_i \cdot \left(d\mathbf{l}' \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\right) \tag{3}$$

and then integrate both sides

$$\frac{4\pi}{\mu_0 I}B_i = \oint \hat{e}_i \cdot \left(d\mathbf{l}' \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\right). \tag{4}$$

To evaluate the integral we need to use

$$\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} = \nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right). \tag{5}$$

The integral becomes

$$\frac{4\pi}{\mu_0 I}B_i = \oint \hat{e}_i \cdot \left(d\mathbf{l}' \times \nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\right) \tag{6}$$

$$= \oint \left(d\mathbf{l}' \cdot \left(\nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) \times \hat{e}_i\right)\right) \tag{7}$$

by the triple product rule $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})$. Then we use Stokes's theorem $\oint_{\partial S}\mathbf{A}\cdot d\mathbf{l} = \int_S(\nabla\times\mathbf{A})\cdot d\mathbf{a}$ to convert this to a surface integral,

$$= \int_S \left(\nabla' \times \left(\nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) \times \hat{e}_i\right)\right) \cdot d\mathbf{a}' \tag{8}$$

Next we simplify the curl of a cross product using $\nabla'\times(\mathbf{a}\times\mathbf{b}) = \mathbf{a}(\nabla'\cdot\mathbf{b}) - \mathbf{b}(\nabla'\cdot\mathbf{a}) + (\mathbf{b}\cdot\nabla')\mathbf{a} - (\mathbf{a}\cdot\nabla')\mathbf{b}$.

$$\nabla' \times \left(\nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) \times \hat{e}_i\right) = \nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)(\nabla'\cdot\hat{e}_i) - \hat{e}_i\left(\nabla'\cdot\nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\right) \tag{9}$$

$$+ (\hat{e}_i\cdot\nabla')\nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) - \left(\nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\cdot\nabla'\right)\hat{e}_i \tag{10}$$

Two of the terms from row one are zero because $\nabla' \cdot \hat{e}_i = 0$ and $(\mathbf{v} \cdot \nabla')\hat{e}_i = 0$, to get

$$= -\hat{e}_i \left(\nabla' \cdot \nabla' \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \right) + (\hat{e}_i \cdot \nabla') \nabla' \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \tag{11}$$

The first remaining term is zero because

$$\begin{aligned} -\hat{e}_i \left(\nabla' \cdot \nabla' \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \right) &= -\hat{e}_i \left(\nabla'^2 \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \right) \\ &= -\hat{e}_i \left(-4\pi \delta \left(\mathbf{x} - \mathbf{x}' \right) \right) \\ &= 4\pi \hat{e}_i \delta \left(\mathbf{x} - \mathbf{x}' \right) \end{aligned} \tag{12}$$

where we used that $1/|x|$ is the Green's function of the Laplace operator. Let us assume that $\mathbf{x} \neq \mathbf{x}'$, i.e. that the point P is not on where the current runs, then the δ-term is zero.
We are left with

$$\begin{aligned} \frac{4\pi}{\mu_0 I} B_i &= \int_S (\hat{e}_i \cdot \nabla') \nabla' \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \cdot d\mathbf{a}' && (13) \\ &= \int_S \left(\frac{\partial}{\partial x_i'} \nabla' \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \right) \cdot d\mathbf{a}' && (14) \\ &= \int_S \left(-\frac{\partial}{\partial x_i} \nabla' \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \right) \cdot d\mathbf{a}' && (15) \\ &= -\frac{\partial}{\partial x_i} \int_S \left(\nabla' \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \right) \cdot d\mathbf{a}'. && (16) \end{aligned}$$

On page 33 in Jackson we find

$$d\Omega' = \frac{\cos(\theta) da'}{|\mathbf{x} - \mathbf{x}'|^2} = -d\mathbf{a}' \cdot \nabla' \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \tag{17}$$

The sign of $d\Omega$ depends on the direction of $\hat{n}$ as stated in the beginning. Altogether

$$\begin{aligned} \frac{4\pi}{\mu_0 I} B_i &= -\frac{\partial}{\partial x_i} \int_S \left(\nabla' \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \right) \cdot d\mathbf{a}' && (18) \\ &= -\frac{\partial}{\partial x_i} \int_S -d\Omega' = \frac{\partial}{\partial x_i} \Omega(\mathbf{x}) && (19) \end{aligned}$$

In vector form this is

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \nabla \Omega(\mathbf{x}). \tag{20}$$

2 Linking Number

Consider two curves in space, $C : x_i(s)$ and $C' : x_i'(s')$. They are circles topologically. Define the linking number m as follows: Deform one of the curves to a circle, and count the number of times the second curve passes through the disk spanned by that circle, counting +1 if it passes in the direction of the normal of the disk and −1 if it passes in the other direction. Use your knowledge of magnetostatics to prove that

$$m = \frac{1}{4\pi} \epsilon_{ijk} \int_C \int_{C'} \frac{\left(x_i - x_i' \right) dl_j dl_k}{|x - x'|^3}. \tag{21}$$

Solution

Gaussian units with $c = 1$ are used here.
To connect these loops to electrostatics / magnetostatics, simply imagine we run a current I through the loop C', meaning

$$j_i\left(x'\right) d^3x' = I d\ell'_i. \tag{22}$$

Then we can use Ampère's law, or Stokes's theorem, which tells us that by integrating $\mathbf{B}$ along a closed curve C, it equals the integral over the enclosed region of a current:

$$\int_{\partial S} d\ell \cdot \mathbf{B} = 4\pi \int_S d\mathbf{S} \cdot \mathbf{j} \tag{23}$$

But of course we know the total current through the area: It's current I times the number the wire C' passes through, $I \cdot m$. So

$$4\pi I m = \int_C d\ell_i B_i \tag{24}$$

Now let's use what we were given in the first exercise, to calculate B:

$$\mathbf{B} = \int_{C'} I \frac{\mathrm{d}\ell \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}. \tag{25}$$

Integrate this over C, and voilà, we're done:

$$m = \frac{1}{4\pi} \epsilon_{ijk} \int_C d\ell_j \int_{C'} d\ell'_k \frac{\left(x_i - x'_i\right)}{|\mathbf{x} - \mathbf{x}'|^3} \tag{26}$$

Note that the previous exercise tells us that this is related to the solid angle. We can make this intuitive, by realizing that, whenever one loop winds through another, that other loop spans a solid angle of 4π. The visualization of this is here: `https://www.wolframcloud.com/obj/d2ecc824-73b9-49fd-8639-8e53730ec324`. I don't know how long that stays online.

3 Aharonov-Bohm

Define a vector potential on a region of space strictly outside the z-axis, such that $\mathbf{A}(\mathbf{x})$ is independent of z, gives a vanishing magnetic field outside the z-axis, and cannot be gauge transformed to zero. Discuss the last point in some detail, and give a physical interpretation. [1]

Solution

We want a vector field defined on $V = \{(x, y, z) \mid (x, y, z) \neq (0, 0, z)\}$ that fulfills

$$\begin{cases} \mathbf{A}(\mathbf{x}) = \mathbf{A}(x, y) \\ \mathbf{B}(\mathbf{x}) = \nabla \times \mathbf{A}(\mathbf{x}) = 0 \\ \mathbf{A}'(\mathbf{x}) = \mathbf{A}(\mathbf{x}) + \nabla\psi(\mathbf{x}) \rightarrow \mathbf{A}' \neq \mathbf{0} \end{cases} \tag{27}$$

From the first two conditions we obtain

$$\begin{aligned} &\mathbf{B}(\mathbf{x}) = \nabla \times \mathbf{A}(\mathbf{x}) = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{z} \\ &\mathbf{A}(\mathbf{x}) = \mathbf{A}(x, y) = \left(A_x(x, y), A_y(x, y), A_z(x, y)\right) \rightarrow \frac{\partial A_z(x, y)}{\partial y} = 0, \frac{\partial A_z(x, y)}{\partial x} = 0 \\ &\mathbf{B}(\mathbf{x}) = 0 \rightarrow \frac{\partial A_y(x, y)}{\partial x} = \frac{\partial A_x(x, y)}{\partial y}. \end{aligned} \tag{28}$$

[1] Previous tutor Nadia recommends: When you are done, consult TT Wu and CN Yang, Concept of non-integrable phase factors and global formulation of gauge fields, Phys. Rev. D12 (1975) 3845 .

Therefore we know A_z must be a constant, $A_z(x, y) = C_z$.
Poincaré's lemma says that a covariant vector field $A_\alpha(x)$ can be written as the gradient of a scalar if and only if its field strength $F_{\alpha\beta}$ is zero. This lemma holds for simply connected spaces, i.e spaces where a closed curve can be deformed to a point without leaving the space. For example, $\mathbb{R}^3$ is simply connected but our space V, which excludes the z-axis is not. In V a closed loop around the z-axis cannot be deformed to a point without leaving the space.
For a vector potential that points around the z-axis the field strength can be zero while the vector potential itself cannot be set to zero via a gauge transformation. Poincaré's lemma does not hold in this case.
Let us try a vector potential that points around the z-axis. In cylindrical polar coordinates we can take

$$\mathbf{A} = \frac{1}{\rho}\hat{\phi} \tag{29}$$

which in regular Cartesian coordinates is

$$\mathbf{A} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right). \tag{30}$$

This has $\nabla \times \mathbf{A} = \mathbf{0}$. From the cylindrical coordinate form we can see that it looks to have the form of a gradient,

$$\begin{aligned}
&\mathbf{A} = \frac{1}{\rho}\hat{\phi} \\
&\nabla f = \frac{\partial f}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\hat{\phi} + \frac{\partial f}{\partial z}\hat{z} \\
&\rightarrow \mathbf{A} = \frac{1}{\rho}\hat{\phi} = \frac{1}{\rho}\frac{\partial f}{\partial \phi}\hat{\phi} \\
&\rightarrow \frac{\partial f}{\partial \phi} = 1 \rightarrow f(\phi) = \phi + c.
\end{aligned} \tag{31}$$

$f(x, y)$ is not well defined on the entire space, since the z-axis is excluded. The problem is that $f(\phi) = f(x, y)$ is not single valued since at, for example, $(x > 0, y = 0)$ we have $f(\phi) = 0$ but if we go one closed curve around to $(x > 0, y = 0)$ then $f(\phi) = 2\pi$. Therefore f is not single valued and does not have a well defined gradient.
This is for the global case. If we consider a connected region, where all closed curves can be deformed to points without leaving the space, in the space then locally the potential can be gauge transformed to zero. But not globally.

A Complementary Notes on Aharonov-Bohm

The Aharonov-Bohm effect is a quantum effect on a charged particle. Say we have a cylinder with a non-zero magnetic field inside but a zero magnetic field outside. We can still have a non-zero vector potential outside, just as we calculated above. The Aharonov-Bohm effect shows that, when considering quantum systems, not only the EM-field strength is enough to describe electromagnetism, also the phase factor

$$\exp\left(\frac{ie}{\hbar c}\oint A_\mu dx^\mu\right) \tag{32}$$

is needed. The phase factor depends on the vector potential, not the magnetic field.
Since this is a quantum effect we need a quantum system to see it. When a charged particle (wave function) travels in a region with zero electromagnetic field but non-zero vector potential the particle is affected by the vector potential and it picks up the phase factor. However, the phase factor does not affect how we observe the wave function, since for this only $|\psi|$ matters. To see the effect we need a quantum phenomenon, such as interference of wave functions.

The Aharonov-Bohm experiment is done by sending a beam of electrons (charged particles) towards a double slit experiment but with a cylinder near the double-slit screen. The cylinder has a non-zero magnetic field inside it, however, outside where the particles travel the field is zero and the potential non-zero. Say we send two of the particles and they take "different paths" and pick up different phase factors. When they interact and form an interference pattern after passing the double-slits we can measure the phase shift (difference between the phase factors) that they picked up.
The interference fringes depend on the phase factor

$$\exp\left(\frac{ie}{\hbar c}\oint A_\mu dx^\mu\right) = \exp\left(-\frac{ie}{\hbar c}\Omega\right) \tag{33}$$

where $\mathbf{\Omega}$ is the flux in the cylinder. Two different fluxes can give the same interference pattern if

$$\Omega_a - \Omega_b = \frac{hc}{e}n \tag{34}$$

where n is an integer.
Say we want to find a gauge transformation from a to b, i.e. $\psi_b = e^{i\alpha}\psi_a$. In terms of the vector potential that is

$$\left(A_\mu\right)_b = \left(A_\mu\right)_a + \frac{\hbar c}{e}\frac{\partial\alpha}{\partial x^\mu}. \tag{35}$$

We can write

$$\Delta\alpha = \frac{e}{\hbar c}\oint\left[\left(A_\mu\right)_b - \left(A_\mu\right)_a\right]dx^\mu = \frac{e}{\hbar c}\left(\Omega_b - \Omega_a\right). \tag{36}$$

If $\Omega_a - \Omega_b = \frac{hc}{e}n$, holds then $\Delta\alpha = 2\pi n$, where n is an integer, and the gauge transformation $S = e^{i\alpha}$ is single valued, which means that a and b can be gauge-transformed into each other. And the effect of the gauge transform cannot be physically measured since the interference pattern is the same. This is as it should be since the gauge choice should not affect the physics, neither classical nor quantum.

B Where does the first formula come from?

How do we get to the formula in Equation 1? Start from Ampère's law,

$$\nabla\times\mathbf{B} = 4\pi\mathbf{J}, \tag{37}$$

and apply another curl to it. The curl of a curl simply is

$$\nabla\nabla\cdot\mathbf{B} - \Delta\mathbf{B} \tag{38}$$

a simple formula, which is further simplified since $\mathbf{B}$ is divergence free. Hence,

$$\Delta\mathbf{B} = -4\pi\nabla\times\mathbf{J}. \tag{39}$$

Now we re-write $\mathbf{J}$ as another Laplacian:

$$\Delta B_i(\mathbf{x}) = -4\pi\nabla\times\mathbf{j}(\mathbf{x}) \tag{40a}$$

$$= -4\pi\nabla\times\int d^3x'\mathbf{j}(\mathbf{x}')\,\delta(\mathbf{x}-\mathbf{x}') \tag{40b}$$

$$= \nabla\times\int d^3x'\mathbf{j}(\mathbf{x}')\left(-4\pi\delta(\mathbf{x}-\mathbf{x}')\right) \tag{40c}$$

$$= \nabla\times\int d^3x'\mathbf{j}(\mathbf{x}')\,\Delta'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) \tag{40d}$$

$$= \nabla\times\int d^3x'\mathbf{j}(\mathbf{x}')\,\Delta\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) \tag{40e}$$

$$= \Delta\left(\nabla\times\int d^3x'\mathbf{j}(\mathbf{x}')\,\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) \tag{40f}$$

which means that

$$B_i(\mathbf{x}) = \epsilon_{ijk}\partial_j \int d^3x' \frac{j_k(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \tag{41}$$

since solutions to the Laplace equation are unique. Finally, evaluate the curl and find

$$B_i(\mathbf{x}) = \epsilon_{ijk}\partial_j \int d^3x' \frac{j_k(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} = \epsilon_{ijk} \int d^3x' \frac{j_k(\mathbf{x}')\left(-\left(x_j - x'_j\right)\right)}{|\mathbf{x}-\mathbf{x}'|^3} \tag{42a}$$

$$= -\epsilon_{ijk} \int d^3x' \frac{j_k(\mathbf{x}')\left(x_j - x'_j\right)}{|\mathbf{x}-\mathbf{x}'|^3} = -\epsilon_{ijk} \int d^3x' \frac{j_k(\mathbf{x}')\left(x_j - x'_j\right)}{|\mathbf{x}-\mathbf{x}'|^3} \tag{42b}$$

$$= \epsilon_{jik} \int d^3x' \frac{j_k(\mathbf{x}')\left(x_j - x'_j\right)}{|\mathbf{x}-\mathbf{x}'|^3} \tag{42c}$$

where we used

$$\partial_j \frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{-\left(x_j - x'_j\right)}{|\mathbf{x}-\mathbf{x}'|^3} \tag{43}$$

and $\epsilon_{ijk} = -\epsilon_{jik}$. Altogether,

$$\mathbf{B}(\mathbf{x}) = \int d^3x' \frac{\mathbf{j}(\mathbf{x}') \times (\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}. \tag{44}$$

Next note that the space integral over the current simplifies, because the current here lives on a wire, so

$$j_i(x')\, d^3x' = I dl'_i \tag{45}$$

to get

$$\mathbf{B}(\mathbf{x}) = I \int_{C'} d\mathbf{l} \times \frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} \tag{46}$$

Acknowledgement

The problems are taken from Ingemar's notes and Jackson's book.
Solutions are adapted from a previous tutor, Nadia Flodgren.