Electrodynamics HT22
Assignment 1

1 Polynomial Orthogonalisation

Compute the first five Legendre polynomials by applying Gram-Schmidt orthogonalization to the polynomials $1, x, x^2, x^3, x^4$.
We first have to understand that the polynomials are a vector space. Mathematically this is straightforward;

$$\mathbb{R}[x] := \operatorname{span}\{x^n \mid n \in \mathbb{N}_0\}, \tag{1}$$

and it is clear that multiplication by $\mathbb{R}$-numbers, and addition of vectors exists and works as expected. To make this more intuitive, you may think of x^n as a meaningless symbol, like the usual basis vectors $\hat{e}_n$, or write the coefficients in a vector, i.e.

$$(c_0, c_1, c_2, \ldots)^T \leftrightarrow c_0 + c_1 x^1 + c_2 x^2 + \ldots. \tag{2}$$

We have defined this vector space through a basis that is natural, x^n, but inconvenient. As you have seen in Ingemar's notes, we care about the integral

$$\int_{-1}^{1} \mathrm{d}x\, P(x)Q(x) =: \langle P, Q \rangle \tag{3}$$

and our basis elements all interfere with one another if we plug a pair of them in here.
We know how to solve this problem as soon as we realize that $\langle ., . \rangle$ is an inner product on this space (discuss in ex.class if unclear). On a vector space with inner product, we can simply follow the Gram-Schmidt procedure to obtain an orthonormal basis. (Countably infinite dimensions don't really make this harder.)
To summarize, we are trying to find a set of polynomials $\{P_n\}$, such that

$$\langle P_m, P_n \rangle = c_n \delta_{m,n} \tag{4}$$

Note that we are not aiming for an ortho*normal* basis, since historically, the basis elements are chosen such that $P_n(1) = 1$, or equivalently the constants above are $c_n = 2/(2n + 1)$.
We build this basis up one dimension at a time. Start with the vector

$$P_0 = (a_0, 0, 0, \ldots)^T \tag{5}$$

in the notation of Equation 2. We see that $a_0 = 1$ is the right normalisation.
In the next step we add in the x^1-subspace. Define a basis vector

$$P_1 = (b_0, b_1, 0, \ldots)\Big(\leftrightarrow b_0 + b_1 x\Big) \tag{6}$$

and find the coefficients by requiring that it must be orthogonal to P_0:

$$0 \stackrel{!}{=} \langle P_0, P_1 \rangle = \int_{-1}^{1} \mathrm{d}x\, (b_0 + b_1 x) = 2b_0 \tag{7}$$

and we see $b_0 = 0$, and $P_1 = (0, b_1, 0, \ldots)^T$. We fix the remaining coefficient using the normalisation condition, $1 = P_1(1) = b_1$, such that

$$P_1 = (0, 1, 0, \ldots). \tag{8}$$

Let's step into the third dimension, where interesting stuff starts to happen, finally. Define

$$P_2 := (c_0, c_1, c_2, 0, \ldots)\Big(\leftrightarrow c_0 + c_1 x + c_2 x^2\Big), \tag{9}$$

and require orthogonality to both P_0 and P_1:

$$0 \overset{!}{=} \langle P_0, P_2 \rangle = \int_{-1}^{1} \mathrm{d}x \, 1 \cdot \left(c_0 + c_1 x + c_2 x^2\right) = 2c_0 + \frac{2}{3} c_2, \tag{10a}$$

$$0 \overset{!}{=} \langle P_1, P_2 \rangle = \int_{-1}^{1} \mathrm{d}x \, x \cdot \left(c_0 + c_1 x + c_2 x^2\right) = \frac{2}{3} c_1, \tag{10b}$$

and we can see the solution

$$P_2 = (-1/2, 0, 3/2, 0, \ldots)^T, \tag{11}$$

where we also again fixed the normalisation $P_2(1) = 1$.
Let's do the same thing in four dimensions. To be a bit smarter this time, we name the constants differently:

$$P_2 \coloneqq D \cdot (d_0, d_1, d_2, 1, 0, \ldots)\Big(\leftrightarrow D \cdot (d_0 + d_1 x + d_2 x^2 + d_3 x^3)\Big), \tag{12}$$

$$0 \overset{!}{=} \langle P_0, P_3 \rangle = \int_{-1}^{1} \mathrm{d}x \, 1 \cdot \left(d_0 + d_1 x + d_2 x^2 + 1 \cdot x^3\right) = 2d_0 + \frac{2}{3} d_2, \tag{13a}$$

$$0 \overset{!}{=} \langle P_1, P_3 \rangle = \int_{-1}^{1} \mathrm{d}x \, x \cdot \left(d_0 + d_1 x + d_2 x^2 + 1 \cdot x^3\right) = \frac{2}{3} d_1 + \frac{2}{5} \tag{13b}$$

$$0 \overset{!}{=} \langle P_2, P_3 \rangle = \int_{-1}^{1} \mathrm{d}x \left(-1/2 + 3/2x^2\right) \cdot \left(d_0 + d_1 x + d_2 x^2 + 1 \cdot x^3\right) = d_2 \cdot (-1/3 + 3/5), \tag{13c}$$

and so

$$P_3 = D(0, -3/5, 0, 1). \tag{14}$$

For the normalisation we see that

$$1 \overset{!}{=} P_3(1) = D(-3/5 + 1), \tag{15}$$

and so

$$P_3 = (0, -3/2, 0, 5/2, \ldots). \tag{16}$$

The same computation goes through for P_4, just with more constants to keep track of. The final solution is

$$P_4 = 1/8(3, 0, -30, 0, 35, 0, \ldots). \tag{17}$$

At the end of exercise 1, I did a recap of why we like Legendre polynomials:

1. Green's functions: Tools to solve linear ODEs for general inhomogeneity. (More mathematically, they are the inverse to the differential operator. That's also a good intuition to have.)
2. For our equation the relevant Green's function is $|x|^{-1}$.
3. In practice it occurs as $|x - x'|^{-1}$, which you can expand as a Taylor series. This makes particular sense if you care about the field far away from a given charge combination. It turns out that a given order of the expansion in the distance, $|x'|/|x|$ in that series comes with a specific angle dependence given by a Legendre polynomial.

This is mainly me re-telling Ingemar's notes and Jackson, so look there for more details.

2 Generating function

Show that you recover the first five Legendre polynomials by expanding the generating function

$$g(t,x) = \left(1 - 2xt + t^2\right)^{-1/2} \tag{18}$$

to fourth order in t.
The hands-on way of doing this is through Taylor expansion of g around $t = 0$,

$$g(t,x) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n g(t,x)}{\partial t^n}\Big|_{t=0} t^n = \sum_{n=0}^{\infty} P_n(x) t^n \tag{19}$$

which, if you compare the terms in the sums, leads to an explicit form of P_n in terms of the n-th partial derivative of g. To low orders we can do this directly,

$$g(t,x) = \left(1 - 2xt + t^2\right)^{-1/2} \tag{20}$$

$$= g(x,0) + \frac{\partial g}{\partial t}(x,0)t + O(t^2) \tag{21}$$

$$= 1 + xt + O(t^2) \tag{22}$$

and we can identify $P_0(x) = 1$ and $P_1(x) = x$.
We can go on like this indefinitely but higher derivatives are messy, so we instead use a recursion relation. This lets us avoid all mention of t. Differentiate Equation 18 with respect to t to find

$$\frac{\partial}{\partial t}\left(1 - 2xt + t^2\right)^{-1/2} = \frac{\partial}{\partial t} \sum_{n=0}^{\infty} P_n(x) t^n \tag{23a}$$

$$\implies \frac{x - t}{\left(1 - 2xt + t^2\right)^{3/2}} = \sum_{n=0}^{\infty} n P_n(x) t^{n-1} \tag{23b}$$

$$\implies \frac{x - t}{\left(1 - 2xt + t^2\right)^{1/2}} = \left(1 - 2xt + t^2\right) \sum_{n=0}^{\infty} n P_n(x) t^{n-1} \tag{23c}$$

Now we use Equation 18 once again to replace the fraction on the l.h.s. with the series containing Legendre polynomials, and find

$$\implies (x - t) \sum_{n=0}^{\infty} P_n(x) t^n = \left(1 - 2xt + t^2\right) \sum_{n=0}^{\infty} n P_n(x) t^{n-1} \tag{23d}$$

$$\implies \sum_{n=0}^{\infty} P_n(x) \left(x t^n - t^{n+1}\right) = \sum_{n=1}^{\infty} n P_n(x) \left(t^{n-1} - 2x t^n + t^{n+1}\right). \tag{23e}$$

Now we only need to compare the coefficients of t^m on the two sides. We find

$$t^0 : P_0 x = P_1 \tag{24}$$

$$t^1 : -P_0 + P_1 x = -2P_1 x + 2P_2 \tag{25}$$

$$t^2 : -P_1 + P_2 x = P_1 - 4P_2 x + 3P_3 \tag{26}$$

$$\vdots$$

$$t^m : -P_{m-1} + P_m x = (m+1)P_{m+1} - 2xmP_m + (m-1)P_{m-1} \tag{27}$$

and the last expression is the recursion relation. Rewriting it slightly, we get

$$(m+1)P_{m+1}(x) = (1 + 2m)xP_m(x) - mP_{m-1}(x). \tag{28}$$

Inserting our expressions for P_0 and P_1 we find

$$P_2 = \frac{1}{2}\left(3xP_1(x) - P_0(x)\right) = \frac{1}{2}\left(3x^2 - 1\right) \tag{29}$$

$$P_3 = \frac{1}{2}\left(5x^3 - 3x\right). \tag{30}$$

We find, just like we did in the previous exercise, the first four Legendre polynomials,

$$P_0 = 1 \tag{31}$$

$$P_1 = x \tag{32}$$

$$P_2 = \frac{1}{2}\left(3x^2 - 1\right) \tag{33}$$

$$P_3 = \frac{1}{2}\left(5x^3 - 3x\right) \tag{34}$$

and we can generate higher-order Legendre polynomials from lower-order ones. Note that we know the relation between derivatives of g and P_n, hence this is also a recursion for these derivatives.

3 Spherical Conductor in Constant Electric Field

Find the electric field outside a spherical conductor placed in a constant electric field in two ways: using the method of images, and using an expansion in spherical harmonics.

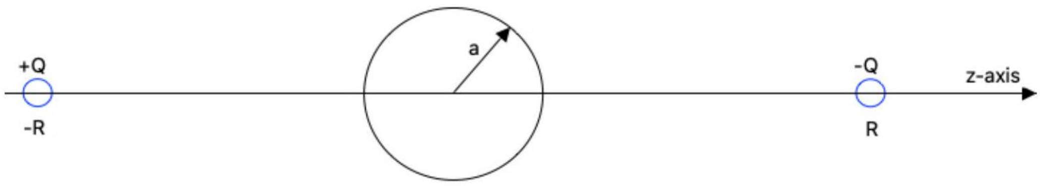

Figure 1: A conducting sphere in a constant electric field

Image Charges Let the constant electric field be in the positive z-direction, $\vec{E} = E_0\hat{z}$. We can construct this field by placing two opposite charges $+Q$ and $-Q$ on the z-axis at distances R from the origin and letting $R \to \infty$. Next, we center the conducting sphere, with radius a, at the origin. At the surface of the sphere the potential is zero, since it is a conducting sphere, and inside it the electric field is zero. To make sure this condition holds we introduce two mirror charges q_1 and q_2 inside the sphere at positions r_1 and r_2. See Fig. 2 . The strength of the electric field is in terms of Q

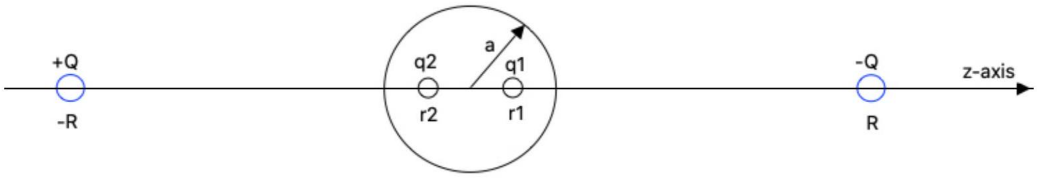

Figure 2: The mirror charges q_1 and q_2 are added to the inside of the sphere. The charges $+Q, -Q$ are there to give the electric field $\bar{E} = E_0\hat{z}$

$$E_0 = \frac{1}{4\pi\epsilon_0}\frac{2Q}{R^2}. \tag{35}$$

We want to find q_1, q_2, r_1, r_2 so that $\Phi(r = a) = 0$. Due to the symmetry of the system we can assume that q_1 and q_2 are of the same strength but opposite charge, $q_1 = q, q_2 = -q$, and that they are symmetrically placed $r_1 = -r_2$. We also note that in spherical coordinates the system has azimuthal symmetry, i.e. it is independent of the angle ϕ.

The potential of a point charge q at position $\bar{x}_0$ is

$$\Phi(\bar{x}) = \frac{1}{4\pi\epsilon_0}\frac{q}{|\bar{x} - \bar{x}_0|}. \tag{36}$$

The potential of the total system is therefore

$$\Phi(\bar{x}) = \frac{1}{4\pi\epsilon_0}\left(-\frac{Q}{|r\hat{r} - R\hat{z}|} + \frac{Q}{|r\hat{r} + R\hat{z}|} + \frac{q}{|r\hat{r} - r_1\hat{z}|} - \frac{q}{|r\hat{r} + r_1\hat{z}|}\right), \tag{37}$$

where $\bar{x} = r\hat{r}$, and $\hat{r}$ is the unit vector in the radial direction, with $r^2 = x^2 + y^2 + z^2$.

We know from Ingemar's notes how we need to place the mirror charges in order to fix a potential on the sphere,

$$\frac{Q}{R} = \frac{q}{a}, \frac{r_1}{a} = \frac{a}{R}. \tag{38}$$

Now we can write $\Phi(\bar{x})$ in spherical coordinates. Next we cheat a bit: Instead of using the two charges $\pm Q$ to write the constant electric field we can use

$$\begin{aligned} \bar{E}(\bar{x}) &= -\nabla\Phi(\bar{x}) \\ \bar{E}(\bar{z}) &= E_0\hat{z} \Rightarrow \Phi(\bar{x}) = -E_0 z = -E_0 r\cos(\theta). \end{aligned} \tag{39}$$

This works because we chose the charges such that later when we push them far away, $R \to \infty$, they will give the constant background field. We don't really evaluate the limit yet for the mirror charges, that's why it's a cheat. We have

$$\begin{aligned}\Phi(\bar{x}) &= -E_0 r\cos(\theta) + \frac{1}{4\pi\epsilon_0}\frac{Qa}{R}\left(\frac{1}{\left|r\hat{r} - \frac{a^2}{R}\hat{z}\right|} - \frac{1}{\left|r\hat{r} + \frac{a^2}{R}\hat{z}\right|}\right) \\ &= -E_0 r\cos(\theta) + \frac{1}{4\pi\epsilon_0}\frac{Qa}{R}\left(\frac{1}{\sqrt{r^2 - \frac{2a^2}{R}r\cos(\theta) + \frac{a^4}{R^2}}} - \frac{1}{\sqrt{r^2 + \frac{2a^2}{R}r\cos(\theta) + \frac{a^4}{R^2}}}\right).\end{aligned} \tag{40}$$

To get the final expression we let $R \to \infty$,

$$\begin{aligned}\lim_{R\to\infty}\Phi(\bar{x}) &\approx -E_0 r\cos(\theta) + \frac{1}{4\pi\epsilon_0}\frac{Qa}{Rr}\left(\sqrt{1 + \frac{2a^2}{Rr}\cos(\theta)} - \sqrt{1 - \frac{2a^2}{Rr}\cos(\theta)}\right) \\ &= -E_0 r\cos(\theta) + \frac{1}{4\pi\epsilon_0}\frac{Qa}{(Rr)^2}2a^2\cos(\theta) \\ &= -E_0 r\cos(\theta) + \frac{E_0 a^3}{r^2}\cos(\theta).\end{aligned} \tag{41}$$

In the first row we removed R^2 terms in the parenthesis, in the second row we used a Taylor expansion and in the final row we wrote Q in terms of E_0

$$Q = \frac{4\pi\epsilon_0 R^2}{2}. \tag{42}$$

The final step is to calculate the electric field, for $r > a$,

$$\begin{aligned}\bar{E}(r,\theta) &= -\nabla\Phi(r,\theta) = -\nabla\left(-E_0 r\cos(\theta) + \frac{E_0 a^3}{r^2}\cos(\theta)\right) \\ &= -\left(\frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta} + \frac{1}{r\sin(\theta)}\frac{\partial}{\partial\phi}\hat{\phi}\right)\left(-E_0 r\cos(\theta) + \frac{E_0 a^3}{r^2}\cos(\theta)\right) \\ &= \hat{r}E_0\cos(\theta)\left(1 + \frac{2a^3}{r^3}\right) - \hat{\theta}E_0\sin(\theta)\left(1 - \frac{a^3}{r^3}\right)\end{aligned} \tag{43}$$

Spherical harmonics

To get the same solution via a direct spherical harmonics calculation, we place the conducting sphere at the origin and once again have the constant electric field in the z-direction, $\bar{E}(\bar{x}) = E_0\hat{z}$. We can use the general axisymmetric solution (see Jackson Eq. (3.33)) of the Laplace equation, $\nabla^2\Phi = 0$, which is

$$\Phi(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos(\theta)), \tag{44}$$

to find the potential. The conditions we want $\Phi(r,\theta)$ to obey are: $\Phi(r = a,\theta) = 0$ and $\Phi(r \to \infty) = -E_0 r\cos(\theta)$. The second condition is that infinitely far away from the sphere the only potential is from the electric field. Using the first condition we find

$$\begin{aligned}\Phi(a,\theta) &= \sum_{l=0}^{\infty}\left(A_l a^l + \frac{B_l}{a^{l+1}}\right)P_l(\cos(\theta)) = 0 \\ &\to A_l a^l + \frac{B_l}{a^{l+1}} = 0 \to B_l = -A_l a^{2l+1}\end{aligned} \tag{45}$$

The potential is now

$$\Phi(r,\theta) = \sum_{l=0}^{\infty}A_l r^l\left(1 - \left(\frac{a}{r}\right)^{2l+1}\right)P_l(\cos(\theta)). \tag{46}$$

Next, we use the second condition

$$\Phi(r,\theta) = -E_0 r\cos(\theta) + o(1) \tag{47}$$

$$A_0\left(1-\frac{a}{r}\right) + A_1 r\left(1-\left(\frac{a}{r}\right)^3\right)\cos(\theta) \tag{48}$$
$$+A_2 r^2\left(1-\left(\frac{a}{r}\right)^5\right)P_2(\cos(\theta)) + \cdots = -E_0 r\cos(\theta) + o(1).$$

This gives $A_0 = 0, A_1 = -E_0, A_l = 0$ for $l \geq 2$, since $P_l(x)$ is proportional to x^l. The final expression for the potential is therefore

$$\Phi(r,\theta) = \frac{E_0 a^3}{r^2}\cos(\theta) - E_0 r\cos(\theta), \tag{49}$$

which is the same as we obtained using the method of images, therefore we will have the same expression for the electric field.

4 Multipoles

Compute the first non-vanishing multipole moments for i) two charges q at $(\pm a, 0, 0)$, charge $-2q$ at $(0,0,b)$ ii) four charges q at $(\pm a, \pm a, 0)$, two charges $-2q$ at $(0,0,\pm b)$. Check your results. The expansion of the potential in terms of the multipole moments is

$$\Phi(\bar{x}) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{r} + \frac{\vec{p}\cdot\vec{x}}{r^3} + \frac{1}{2}\sum_{i,j} Q_{ij}\frac{x_i x_j}{r^5} + \ldots\right] \tag{50}$$

$$= \frac{1}{\epsilon_0}\sum_{l=0}^{\infty}\left(\sum_{m=-l}^{l}\frac{q_{lm}}{2l+1}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}}\right). \tag{51}$$

Here, the last row is the general formulation in terms of spherical harmonics Y_{lm}. Since the charges we are given here are not spherically symmetric, we will mostly use the Cartesian formulation in terms of $q, \vec{p}, Q_{i,j}$. These multipole moments are given by

$$q = \int \mathrm{d}^3x\, \rho(\vec{x}) \tag{52a}$$

$$\vec{p} = \int \mathrm{d}^3x\, \rho(\vec{x})\vec{x} \tag{52b}$$

$$Q_{i,j} = \int \mathrm{d}^3x\, \rho(\vec{x})\left(3x_i x_j - r^2\delta_{ij}\right) \tag{52c}$$

These integrals simplify further for a charge distribution that consists of a collection of discrete point charges, $\rho(\bar{x}) = \sum_{k=1}^{N} q_k \delta\left(\bar{x} - \bar{x}_k\right)$:

$$q = \sum_k q_k \tag{53a}$$

$$\bar{p} = \sum_k q_k \bar{x}_k \tag{53b}$$

$$Q_{i,j} = \sum_k q_k\left(3\left(x_k\right)_i\left(x_k\right)_j - r^2\delta_{ij}\right), \tag{53c}$$

where $r^2 = x^2 + y^2 + z^2$ and $i, j = 1,2,3$ (corresponding to x, y, x).

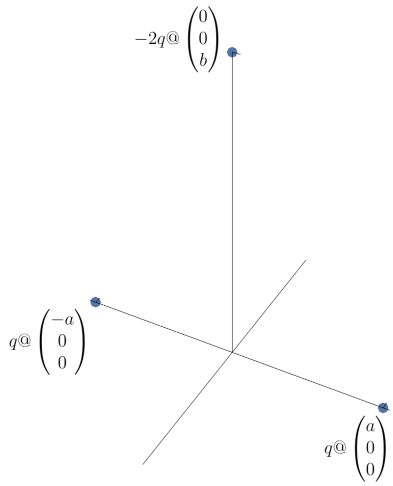

Figure 3: The three point charges considered in exercise 4 part i: Two charges q at $(\pm a, 0, 0)$ and one charge $-2q$ at $(0, 0, b)$.

a) I show the first setup in Figure 3. The monopole moment is

$$q = \sum_i q_i = q + q - 2q = 0. \tag{54}$$

The dipole moment is

$$\bar{p} = \sum_i q_i \bar{x}_i = q(a, 0, 0) + q(-a, 0, 0) - 2q(0, 0, b) = (0, 0, -2qb). \tag{55}$$

Therefore, the dipole moment is the first non-vanishing multipole moment. To check if this is correct we insert the $\bar{p}$ into the potential and then check if the Laplace equation $\nabla^2\Phi(\bar{x}) = 0$ holds. The potential is to its lowest order

$$\Phi(\bar{x}) = \frac{1}{4\pi\epsilon_0}\frac{\bar{p}\cdot\bar{x}}{r^3} = \frac{-2qb}{4\pi\epsilon_0}\frac{z}{r^3}. \tag{56}$$

Inserting this into the Laplace equation we find

$$\nabla^2\Phi(\bar{x}) = \nabla^2\left(\frac{-2qb}{4\pi\epsilon_0}\frac{z}{r^3}\right) = \frac{-2qb}{4\pi\epsilon_0}\left(\frac{\partial^2}{\partial_x^2} + \frac{\partial^2}{\partial_y^2} + \frac{\partial^2}{\partial_z^2}\right)\left(\frac{z}{r^3}\right) = \cdots = 0. \tag{57}$$

We also know that since the dipole moment was not zero the dipole would turn in the presence of a uniform electric field. If we consider the charge distribution it is clear that it has a positive and a negative side, which means it would turn in an electric field to be parallel to that field.

b) See Figure 4 for the setup.

The monopole moment is

$$q = \sum_i q_i = 4q + 2(-2q) = 0. \tag{58}$$

The dipole moment is

$$\begin{aligned}\bar{p} &= q(a, a, 0) + q(a, -a, 0) + q(-a, a, 0) + q(-a, -a, 0) - 2q(0, 0, b) - 2q(0, 0, -b)\\ &= (0, 0, 0).\end{aligned} \tag{59}$$

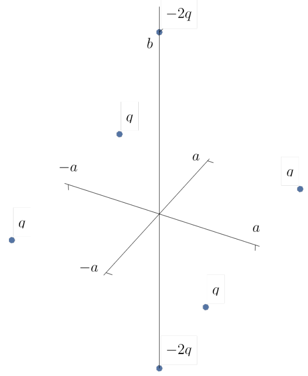

Figure 4: The six point charges in the second part of exercise 4. Four charges q at $(\pm a, \pm a, 0)$, two charges $-2q$ at $(0, 0, \pm b)$.

The quadrupole moment tensor is computed one element at a time.

$$
\begin{aligned}
Q_{ij} &= \sum_k q_k \left(3\,(x_k)_i\,(x_k)_j - r^2\delta_{ij}\right)\\
Q_{12} &= \sum_k q_k 3\,(x_1)_i\,(x_2) = 3\left(qa^2 + q\left(-a^2\right) + q\left(-a^2\right) + q\left(a^2\right)\right) = 0\\
Q_{13} &= 0\\
Q_{23} &= 0\\
Q_{11} &= \sum_k q_k \left(3\left((x_k)_1\right)^2 - \left(x_1^2 + x_2^2 + x_3^2\right)\right) = 4q\left(3a^2 - 2a^2\right) + 2(-2q)\left(-b^2\right) = 4q\left(a^2 + b^2\right)\\
Q_{22} &= 4q\left(a^2 + b^2\right)\\
Q_{33} &= 4q\left(-2a^2\right) + 2(-2q)\left(2b^2\right) = -8q\left(a^2 + b^2\right).
\end{aligned}
\tag{60}
$$

To check if this is correct we insert $\bar{p} = 0$ and Q_{ij} into the potential expansion

$$
\Phi(\bar{x}) = \frac{1}{4\pi\epsilon_0}\frac{1}{2}\sum_{i,j} Q_{ij}\frac{x_i x_j}{r^5} = \frac{1}{4\pi\epsilon_0}\frac{1}{2}q\left(a^2 + b^2\right)\frac{4x^2 + 4y^2 - 8z^2}{r^5} \tag{61}
$$

and test this potential in the Laplace equation, which turns out to be

$$
\nabla^2\Phi(\bar{x}) = \nabla^2\left(\frac{1}{4\pi\epsilon_0}\frac{1}{2}q\left(a^2 + b^2\right)\frac{4x^2 + 4y^2 - 8z^2}{r^5}\right) = \cdots = 0. \tag{62}
$$

This charge distribution had no dipole moment, since $\bar{p} = \overline{0}$, which means in an electric field there would be no torque applied to it. Looking at the positions of the charges in the charge distribution we can see that, unlike in case i), there is no clear positive and negative side to the distribution.

Recall also that the first non-vanishing multipole moment is invariant (under shifts of the origin only of course).

5 Building Blocks for Moments

You have a supply of point charges $\pm q$. We are given positions in which we may put charges, and we try to make the monopole, dipole, quadrupole moments vanish.

8-hedron A regular octahedron has 6 corners, see Figure 5. One way of placing the octahedron, with side length a, in a coordinate system puts the corners at positions:

$$(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1) \tag{63}$$

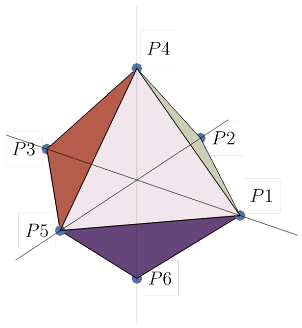

Figure 5: A regular octahedron

For the monopole moment to be zero we require

$$q = \sum_k q_k = q_1 + q_2 + q_3 + q_4 + q_5 + q_6 = 0 \tag{64}$$

where q_k is the charge in corner k. We see that we any configuration of equally many $+q$ and $-q$ charges satisfies this.

The dipole moment is

$$\begin{aligned} \bar{p} &= \sum_k q_k \bar{x}_k \\ &= q_1(1,0,0) + q_2(0,1,0) + q_3(-1,0,0) \\ &\quad + q_4(0,-1,0) + q_5(0,0,1) + q_6(0,0,-1) \\ &= (q_1 - q_3, q_2 - q_4, q_5 - q_6) \,. \end{aligned} \tag{65}$$

For that to be zero we need $q_1 = q_3, q_2 = q_4$ and $q_5 = q_6$, but this would give the monopole moment $q = 2q_1 + 2q_2 + 2q_5$. Thus q cannot be zero if we want $\bar{p} = \bar{0}$. This makes sense, since if we have the same number of corners with $+q$ and $-q$, as required by $q = 0$, the system will move in a constant electric field no matter how you place the charges, and therefore have a dipole moment.

Sneaky surprise twist: We could salvage this by choosing to not have any charges in one direction!

For the quadrupole, let's use instead the $q_{2,m}$ form of the multipole expansion. We find

$$q_{2,m} = \begin{bmatrix} (\text{factor} \cdot (q_1 + q_2 - q_3 - q_4)) \\ 0 \\ (\text{factor} \cdot (q_1 + q_2 + q_3 + q_4 - 2q_5 - 2q_6)) \\ 0 \\ (\text{factor} \cdot (q_1 + q_2 - q_3 - q_4)) \end{bmatrix}_m \tag{66}$$

To make quadrupoles vanish, we must cancel all square formations of charges.

Pairs of opposing charges have no quadrupole and no monopole, but full dipole.

All charges being the same has no dipole, and no quadrupole, but of course full monopole.

Can we gain something by removing charges? No, removing a pair of charges doesn't help in killing all moments simultaneously.

Cube For the cube (Figure 6), with side length $2a$, we can place charges at all eight possible combinations of signs in $(\pm a, \pm a, \pm a)$.

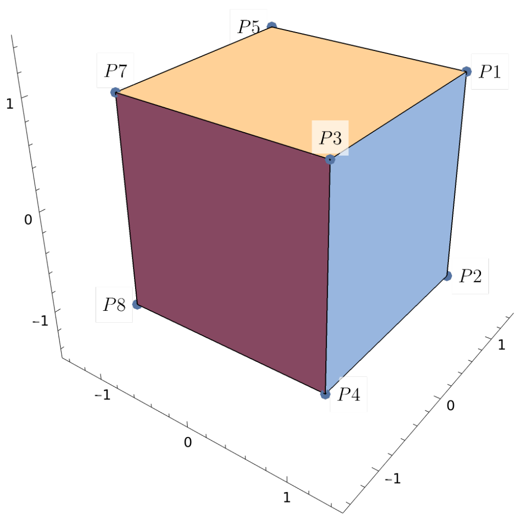

Figure 6: In case you forgot what a cube looks like.

Setting the monopole to zero we get the condition

$$q = q_1 + q_2 + q_3 + q_4 + q_5 + q_6 + q_7 + q_8 = 0. \tag{67}$$

Calculating the dipole

$$\begin{aligned}\bar{p} = \frac{a}{2}\,(&q_1 - q_2 - q_3 + q_4 + q_5 - q_6 - q_7 + q_8,\\ &q_1 + q_2 - q_3 - q_4 + q_5 + q_6 - q_7 - q_8,\\ &-q_1 - q_2 - q_3 - q_4 + q_5 + q_6 + q_7 + q_8)\,.\end{aligned} \tag{68}$$

Setting $\bar{p} = \overline{0}$ we get the system of equations

$$\begin{aligned}q_1 - q_2 - q_3 + q_4 + q_5 - q_6 - q_7 + q_8 &= 0\\ q_1 + q_2 - q_3 - q_4 + q_5 + q_6 - q_7 - q_8 &= 0\\ -q_1 - q_2 - q_3 - q_4 + q_5 + q_6 + q_7 + q_8 &= 0\end{aligned} \tag{69}$$

which combined with the condition $q = 0$ is solved by

$$\begin{aligned}q_4 &= -q_1 - q_2 - q_3\\ q_6 &= -q_1 - q_2 - q_5\\ q_7 &= q_1 - q_3 + q_5\\ q_8 &= q_2 + q_3 - q_5\end{aligned} \tag{70}$$

These equations hold, for example, if each corner has the opposite charge of its neighbors. So it is possible to have $\bar{p} = \overline{0}$ and $q = 0$ for this system in the shape of a cube. The system, with this placement of charges, does not "look like a dipole" since each $+q$ has a $-q$ opposite and beside it.

The quadrupole moment is

$$Q_{ij} = \sum_k q_k \left(3\,(x_k)_i\,(x_k)_j - (r_k)^2\,\delta_{ij}\right), \tag{71}$$

where $r_k = \sqrt{3}a/2$ for all k. Calculating the quadrupole moment we obtain

$$\begin{aligned}
Q_{11} &= Q_{22} = Q_{33} = 0\\
Q_{12} &= 3\left(\frac{a}{2}\right)^2 (q_1 - q_2 + q_3 - q_4 + q_5 - q_6 + q_7 - q_8)\\
Q_{23} &= 3\left(\frac{a}{2}\right)^2 (-q_1 - q_2 + q_3 + q_4 + q_5 + q_6 - q_7 - q_8)\\
Q_{13} &= 3\left(\frac{a}{2}\right)^2 (-q_1 + q_2 + q_3 - q_4 + q_5 - q_6 - q_7 + q_8)\,.
\end{aligned} \tag{72}$$

Setting them equal to zero we find that they can all be fulfilled if

$$\begin{aligned}
q_6 &= q_3 - q_4 + q_5\\
q_7 &= -q_1 + q_3 + q_5\\
q_8 &= -q_2 + q_3 + q_5.
\end{aligned} \tag{73}$$

Combining these equations with the ones for the vanishing dipole moment, eqs. (55), we find that

$$\begin{aligned}
&q_1 = -q_2 = q_3 = -q_4\\
\rightarrow\ &q_1 = -q_5 = q_6 = -q_7 = q_8.
\end{aligned} \tag{74}$$

In this configuration each corner has the opposite charge as its neighboring corners. Therefore, we can have $q = 0, \bar{p} = \overline{0}$ and $Q_{ij} = 0$.

What this exercise is meant to illustrate is that only the first non-vanishing multipole moment is in dependent of the choice of origin. As an example let us shift the origin of the dipole moment and see what happens. We take $\rho(\bar{x})$ and change it to $\rho\,(\bar{x} + \bar{x}_0)$, then the dipole moment $\bar{p}$ changes to

$$\begin{aligned}
\bar{p}_{\text{new}} &= \int \bar{x}\rho\,(\bar{x} + \bar{x}_0)\,dV = \int (\bar{x} - \bar{x}_0)\,\rho(\bar{x})dV\\
&= \int \bar{x}\rho(\bar{x})dV - \bar{x}_0 \int \rho(\bar{x})dV = \bar{p} - \overline{x_0}q.
\end{aligned} \tag{75}$$

Here we see that if the monopole moment is zero the dipole moment is unchanged. If $q \neq 0$ then $\bar{p}_{\text{new}}$ depends on the location $\bar{x}_0$ of the dipole moment.

6 The Dipole Field

For a dipole field, locate those points in space where the field points in a direction orthogonal to the dipole vector.

The potential of a dipole $\bar{p}$ is

$$\Phi(\bar{x}) = \frac{1}{4\pi\epsilon_0}\frac{\bar{p}\cdot\bar{x}}{r^3} \tag{76}$$

and to find its electric field we can simply apply $\bar{E}(\bar{x}) = -\bar{\nabla}\Phi(\bar{x})$. The electric field at $\bar{x}$ of a dipole at $\bar{x}_0$ is

$$\bar{E}(\bar{x}) = \frac{1}{4\pi\epsilon_0}\frac{3\bar{n}(\bar{p}\cdot\bar{n}) - \bar{p}}{|\bar{x} - \bar{x}_0|^3} \tag{77}$$

where $\bar{n}$ is a unit vector from $\bar{x}_0$ to $\bar{x}$. Let us place the dipole at the origin, $\bar{x}_0 = \overline{0}$, then $\bar{n} = \frac{\bar{x}}{|\bar{x}|} = \frac{\bar{x}}{r}$, where $|\bar{x}| = r$.
We want to find the points where the dipole vector and its electric field are orthogonal to each other, i.e. where $\bar{E}(\bar{x}) \cdot \bar{p} = 0$. Therefore, we evaluate $\bar{E}(\bar{x}) \cdot \bar{p}$ and set it to zero.

$$\begin{aligned}
&\bar{E}(\bar{x}) \cdot \bar{p} = \frac{1}{4\pi\epsilon_0} \frac{3\bar{n}(\bar{p}\cdot\bar{n}) - \bar{p}}{r^3} \cdot \bar{p} = 0 \\
&\rightarrow (3\bar{n}(\bar{p}\cdot\bar{n}) - \bar{p}) \cdot \bar{p} = 0 \\
&\rightarrow (3\bar{n}(\bar{p}\cdot\bar{n}) - \bar{p}) \cdot \bar{p} = \frac{3}{r^2}\bar{x}(\bar{p}\cdot\bar{x}) \cdot \bar{p} - \bar{p}^2 = \frac{3}{r^2}(\bar{p}\cdot\bar{x})^2 - \bar{p}^2 \\
&= \frac{3(\bar{p}\cdot\bar{x})^2 - r^2\bar{p}^2}{r^2} = 0 \\
&\rightarrow 3(\bar{p}\cdot\bar{x})^2 - r^2\bar{p}^2 = 0.
\end{aligned} \tag{78}$$

Expanding the last line we find

$$\begin{aligned}
3(\bar{p}\cdot\bar{x})^2 - r^2\bar{p}^2 &= 3\left(xp_x + yp_y + zp_z\right)^2 - \left(x^2+y^2+z^2\right)\left(p_x^2+p_y^2+p_z^2\right) \\
&= p_x^2\left(2x^2 - y^2 - z^2\right) + p_y^2\left(2y^2 - x^2 - z^2\right) + p_z^2\left(2z^2 - x^2 - y^2\right) \\
&+ 6\left(p_xp_yxy + p_xp_zxz + p_yp_zyz\right) = 0.
\end{aligned} \tag{79}$$

This is zero if two of the p_i are zero. Let us specify the direction of the dipole, say $\bar{p} = p_z\hat{z}$. Inserting $p_x = p_y = 0$ into the above equation gives us

$$2z^2 = x^2 + y^2 \tag{80}$$

This is the equation of a cone centered on the z-axis. I show this cone in Figure 7.

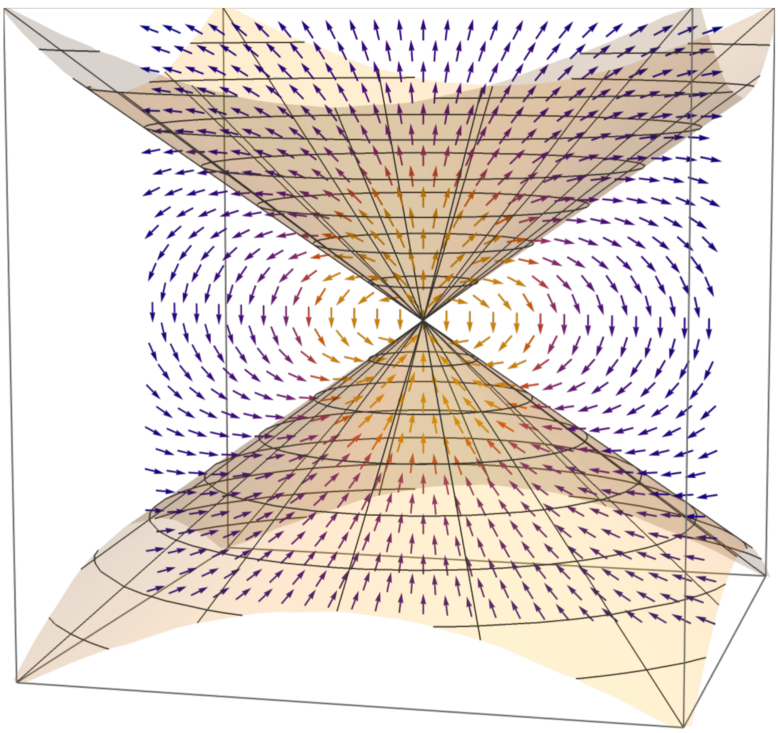

Figure 7: Plot of the electric field around a dipole, shown as arrows on a plane containing $\bar{p}$. On the cone shown in the figure, the field is orthogonal to the dipole moment. The field and the cone are rotationally symmetric around the dipole axis.

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