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Covariant Superstrings Do Not Admit Covariant Gauge Fixing

by

Ingemar Bengtsson and Martin Cederwall

Abstract

The covariant action for superstrings is analyzed by means of the Dirac method, and the light front action is derived. The action gives rise to second class constraints which cannot be solved for in a covariant way. Covariant gauge fixing, in any conventional sense, cannot be applied, which leaves the question of covariant quantization of the superstrings open. A similar statement holds for a massless supersymmetric point-particle.

1. Introduction

The subject of superstrings [1] has been developed by Green and Schwarz. In Ref. [2] they presented a light front action which describes the quantum mechanics of free superstrings. For closed boundary conditions this action gives a superstring with N=2 supersymmetry (in ten space-time dimensions). A covariant formulation of this interesting model - it is connected to N=8 supergravity in the so-called zero-slope limit of the interacting theory - is lacking, though. In Ref. [3], a covariant action was given, which in a light front gauge reduces to the action of Ref. [2]. We will demonstrate that the covariant action does not admit covariant gaug fixing. Before the completion of this work, a paper by Green and Schwarz [4] appeared in which a covariant gauge was proposed. Our analysis shows that their procedure is deficient.

The reason why covariant gauge fixing is impossible is to be found in the constraint structure of the model. For bosonic strings, the covariant action gives rise to constraints among the canonical variables which are all first class, i.e. they obey a closed algebra among themselves and with the Hamiltonia (the terminology is that of Dirac [5]). First class constraint correspond to gauge symmetries. They may be imposed as conditions on the quantum states, while the naive Poisson brackets are retained. This strategy is used in covariant quantization, most elegantly performed using the BRS technique (see Ref. [6] for BRS quantization of bosonic strings). The superstring acti gives rise to some constraints which do not obey the above men tioned condition. Such constraints are called second class and they have to be explicitly accounted for before quantization is attempted. In the canonical formalism the naive Poisson brackets have to be replaced by Dirac brackets;

$$[A,B]^{*}=[A,B]-[A,\phi_i][\phi_i,\phi_j]^{-1}[\phi_j,B]. \qquad (1)$$

Here, the ϕ_i's are second class constraints. In the path integral formalism, the constraints give rise to a non-trivial factor in the path integral measure [7], as follows:

$$\Pi_i\,\delta(\chi_i)\,\delta(\psi_i)\,\det[\chi_i,\psi_i]\,\Pi_k\,\delta(\phi_k)\sqrt{\det[\phi_k,\phi_l]} \qquad (2)$$

where ψ_i are first class constraints and χ_i are gauge conditions. The problem that we will encounter in the superstring model is that the second class constraints cannot be accounted for in a covariant way (in fact, they cannot even be identified in a covariant way).

There exists a supersymmetric point particle action [8] which is closely analogous to the string actions discussed here, and we will begin by analyzing this simpler case. See the appendix for our conventions.

2. A supersymmetric point particle

The supersymmetric point particle action is

$$S=\frac{1}{2}\int d\tau\,[\,\frac{1}{V}\dot{Z}^2+Vm^2\,]\,, \qquad (3)$$

where

$$\dot{Z}^\mu=\dot{x}^\mu-i\bar{\theta}\gamma^\mu\dot{\theta}\,. \qquad (4)$$

The action is left invariant by the global supersymmetry transformation

$$\delta x^\mu=i\bar{\varepsilon}\gamma^\mu\theta\,,\qquad \delta\theta^a=\varepsilon^a\,,\qquad \delta V=0\,. \qquad (5)$$

The momenta conjugate to x^μ, $\bar{\theta}_a$ and V are

$$P_\mu=\frac{1}{V}\dot{Z}_\mu\,,\qquad p_\theta^a=\frac{i}{V}(\gamma^\mu\theta)^a\dot{Z}_\mu\,,\qquad P_v=0\,. \qquad (6)$$

We recognize the primary constraints

$$\phi^a\equiv p_\theta^a-i(\gamma^\mu\theta)^a P_\mu=0\,,\qquad P_v=0\,. \qquad (7)$$

The canonical Hamiltonian may now be calculated,

$$H_c=\dot{x}^\mu P_\mu+\dot{\bar{\theta}}p_\theta+\dot{V}P_v-\mathcal{L}=\frac{1}{2}V(P^2+m^2)\,. \qquad (8)$$

We impose on the coordinates and their conjugate momenta the naive Poisson brackets at equal τ

$$[x^\mu, P^\nu] = \eta^{\mu\nu} ,$$
$$\{\theta^a, \bar{P}_{\theta b}\} = \delta^a_b , \qquad [V, P_V] = 1 , \tag{9}$$

and all other Poisson brackets zero.

We now have to ensure that the constraints are consistent with the time evolution of the system, e.g. we have to require that

$$[P_V, H] = 0 . \tag{10}$$

This generates a secondary constraint

$$\psi \equiv P^2 + m^2 = 0 . \tag{11}$$

It follows that H is weakly zero, in the sense of Dirac. Having found the secondary constraint, it is convenient to impose the gauge choice V=1, after which V and P_V may be dropped from the problem.

The naive Poisson brackets among the constraints are

$$[\phi^a, \phi^b] = 2i(\gamma_\mu C)^{ab} P^\mu , \tag{12}$$

and all the others vanish. Thus, ψ is a first class constraint, which corresponds to reparametrization invariance. In the massive case, all the ϕ^a's are second class, since the matrix in Eq. (12) possesses the inverse

$$[\phi^a, \phi^b]^{-1} = \frac{i}{2}\frac{(C\gamma_\mu)_{ab}P^\mu}{P^2} = -\frac{i}{2}\frac{(C\gamma_\mu)_{ab}P^\mu}{m^2} . \tag{13}$$

It is straight forward to introduce the appropriate Dirac brackets [8].

When m=0, the matrix in Eq. (12) becomes singular and th inverse no longer exists. This means that some of the ϕ^a's ar first class, corresponding to an extra gaugesymmetry of the a tion (the local supersymmetry of ref. [9]).

In fact, the ϕ^a's contain both first and second class co straints. Since first and second class constraints are to be handled in different ways, it is <u>necessary</u> to separate them from each other before we proceed. However, the required sepa ration <u>cannot</u> be performed covariantly. This is clear on dime sional grounds - the constraint matrix (12) has dimension of mass, so that it is necessary to divide by an object having dimension of mass in order to invert any part of it. Unless m ifest covariance is given up, the only available object is P which is zero. Therefore the second class constraints cannot eliminated in a covariant fashion.

A non-covariant decomposition may be performed, however In particular, the light front decomposition

$$\phi = \phi_+ + \phi_- = \tfrac{1}{2}\gamma_-\gamma_+\phi + \tfrac{1}{2}\gamma_+\gamma_-\phi \tag{14}$$

proves to serve our purposes well, ϕ_- may be viewed as secon class since

$$[\phi_-^a, \phi_-^b] = 2i(\gamma_+ C)^{ab} P^+ , \tag{15}$$

the determinant of which is zero only for $P^+=0$. This singularity is not serious (more over, it is unavoidable for massless particles [10]). We then find the Dirac brackets

$$\{\phi_+^a, \phi_+^b\}^* = i\frac{P^2}{P^+}(\gamma^+ C)^{ab} \quad . \tag{16}$$

Hence, the ϕ_+^a's have become first class constraints. They allow the gauge choice $\theta_-=0$.

The essential points of our reasoning should be clear from this example. The superstring case involves slightly more work but no new ideas.

3. Generalization to Superstrings

A natural generalization of the point particle action - first considered in 1976 [11] - appears to be

$$S = \frac{1}{2}\int d\tau d\sigma (-g)^{1/2} g^{\alpha\beta} Z^\mu_\alpha Z_{\mu\beta} \quad , \tag{17}$$

where

$$Z^\mu_\alpha = \partial_\alpha X^\mu - i\bar\theta\gamma^\mu\partial_\alpha\theta \quad . \tag{18}$$

However, it exhibits undesirable features. Let us sketch what they are. In the ON-gauge $g^{\alpha\beta}=\eta^{\alpha\beta}$ (corresponding to the gauge V=1 in the point particle case) one finds the first class constraints (corresponding to the reparametrization invariance of the world sheet of the string)

$$\begin{aligned} \psi_1(\sigma) &\equiv P^2 + Z'^2 = 0 \quad , \\ \psi_2(\sigma) &\equiv P^\mu Z'_\mu = 0 \quad , \end{aligned} \tag{19}$$

as well as the spinorial constraints

$$\phi^a(\sigma) \equiv p_\theta^a - i(\gamma_\mu\theta)^a P^\mu = 0 \quad . \tag{20}$$

Using the naive Poisson brackets, one finds that

$$\{\phi^a(\sigma), \phi^b(\sigma')\} = -2i(\gamma_\mu C)^{ab} P^\mu \delta(\sigma-\sigma') \quad . \tag{21}$$

Since the determinant of this matrix is non-zero in general, there are no first class constraints among the ϕ^a's, and thus no gauge symmetry to allow the gauge choice $\theta_-=0$. For this reason the action (17) cannot be used to describe the superstrings of ref. [2]. On the other hand, the matrix (21) is singular when $P^2(\sigma)=0$. We believe that this singularity is serious and means that the model is inconsistent. However, it is concievable that the action (17) could be used in connection with a Polyakov-type string in four dimensions, having a massive spectrum, if supplemented with some extra degrees of freedom (although it should be said that definite obstacles to a Polyakov treatment of superstrings are known [12]). Note that the supercharge

$$Q^a = \int d\sigma\, [p_0^a + i(\gamma_\mu\theta)^a P^\mu] = 2i\int d\sigma\, (\gamma_\mu\theta)^a P^\mu \tag{22}$$

contains enough independent degrees of freedom to create an N=1 massive supermultiplet (which again shows that the action (17) has nothing to do with massless superstrings).

In order to make the action (17) locally supersymmetric, i.e. to make some of the spinorial constraints first class, some extra terms have to be added. The action must contain an N=2 symmetry and was obtained in ref. [3],

$$\begin{aligned} S = \int d\tau d\sigma \{ & -\tfrac{1}{2}(-g)^{1/2} g^{\alpha\beta} Z^\mu_\alpha Z_{\mu\beta} - \\ & - i\varepsilon^{\alpha\beta}\partial_\alpha X^\mu(\bar\theta^1\gamma_\mu\partial_\beta\theta^1 - \bar\theta^2\gamma_\mu\partial_\beta\theta^2) + \\ & + \varepsilon^{\alpha\beta}\bar\theta^1\gamma^\mu\partial_\alpha\theta^1\bar\theta^2\gamma_\mu\partial_\beta\theta^2 \} . \end{aligned} \tag{23}$$

The emanating constraints become, in close analogy with (7), (11) and (19), (20):

$$\psi_1 \equiv (P+S)^2 + Z'^2 = 0 , \tag{24}$$

$$\psi_2 \equiv 2(P^\mu+S^\mu)Z'_\mu = 0 , \tag{25}$$

$$\phi^{1a} \equiv p_0^{1a} - i(\gamma_\mu\theta^1)^a(P^\mu - X'^\mu + i\bar\theta^1\gamma^\mu\theta'^1) = 0 , \tag{26}$$

$$\phi^{2a} \equiv p_0^{2a} - i(\gamma_\mu\theta^2)^a(P^\mu + X'^\mu - i\bar\theta^2\gamma^\mu\theta'^2) = 0 , \tag{27}$$

where

$$S^\mu = i(\bar\theta^1\gamma^\mu\theta'^1 - \bar\theta^2\gamma^\mu\theta'^2) . \tag{28}$$

The naive Poisson bracket relations among the constraint calculated for the case of closed strings, are

$$\begin{aligned} [\psi_1(\sigma),\psi_1(\sigma')] &= 2(\psi_2(\sigma)+\psi_2(\sigma'))\partial_\sigma\delta(\sigma-\sigma') , \\ [\psi_1(\sigma),\psi_2(\sigma')] &= 2(\psi_1(\sigma)+\psi_1(\sigma'))\partial_\sigma\delta(\sigma-\sigma') , \\ [\psi_2(\sigma),\psi_2(\sigma')] &= 2(\psi_2(\sigma)+\psi_2(\sigma'))\partial_\sigma\delta(\sigma-\sigma') , \end{aligned} \tag{29}$$

$$\{\phi^{Aa}(\sigma),\phi^{Bb}(\sigma')\} = 2i\delta^{AB}(\gamma_\mu C)^{ab}\Pi_A^\mu(\sigma)\delta(\sigma-\sigma') , \tag{30}$$

$$\begin{aligned} [\phi^{Aa}(\sigma),\psi_2(\sigma')] &= (-1)^A[\phi^{Aa}(\sigma),\psi_1(\sigma')] = \\ &= 4i(\gamma_\mu\theta'^A)^a(\sigma)\Pi_A^\mu(\sigma)\delta(\sigma-\sigma') , \end{aligned} \tag{31}$$

(A is not summed over), where

$$\Pi^{\mu}_{\frac{1}{2}} = P^{\mu} + S^{\mu} \mp \acute{Z}^{\mu} . \tag{32}$$

Note that

$$(\Pi_{\frac{1}{2}})^2 = \psi_1 \mp \psi_2 \quad . \tag{33}$$

An attempt to invert the constraint matrix $C^{ABab}(\sigma,\sigma')$ of Eq. (30) gives the result

$$(C^{-1})^{AB}_{ab}(\sigma,\sigma') = \frac{i}{2}\delta^{AB}(C\gamma_{\mu})_{ab}\frac{\Pi^{\mu}_{A}(\sigma)}{(\Pi_A)^2(\sigma)}\delta(\sigma-\sigma') \tag{34}$$

This cannot be allowed since division by zero has been performed (this is similar to the case of the massless point particle). However, the analysis of ref. [4] amounts to this. The authors of ref. [4] proceed to quantize the model using the ensuing "Dirac brackets", and fail to do so which is a consequence of their inconsistent classical treatment. Another serious mistake in treating all the spinorial constraints as second class, as is done in ref. [4], is that one miscounts the number of degrees of freedom. It is clear that - as in the massless point particle case - the spinorial constraints contain both first and second class constraints, which have to be disentangled before we proceed. A strong argument against this problem having a covariant solution, is the fact that, since the ϕ^A's already are in the smallest (16-dimensional) spinor representation of SO(1,9), they cannot be decomposed covariantly. However, this argument has a flaw, to which we will return. A fruitful choice for a non-covariant decomposition is the light front decomposition

$$\phi = \phi_+ + \phi_- = \frac{1}{2}\gamma_-\gamma_+\phi + \frac{1}{2}\gamma_+\gamma_-\phi . \tag{35}$$

ϕ_-^A may be viewed as second class, since

$$\{\phi_-^{Aa}(\sigma),\phi_-^{Bb}(\sigma')\} = 2i\delta^{AB}(\gamma_+C)^{ab}\,\Pi_A^+(\sigma)\,\delta(\sigma-\sigma') , \tag{36}$$

the determinant of which is zero only for $\pi_A^+(\sigma)=0$, the usual singularity of the light front formalism. The ensuing Dirac brackets between the remaining constraints are

$$[\psi_i(\sigma),\psi_j(\sigma')]^* = [\psi_i(\sigma),\psi_j(\sigma')] , \tag{37}$$

$$\{\phi_+^{Aa}(\sigma),\phi_+^{Bb}(\sigma')\}^* = i\delta^{AB}(\gamma_-C)^{ab}\frac{(\Pi_A)^2(\sigma)}{\Pi_A^+(\sigma)}\delta(\sigma-\sigma') , \tag{38}$$

$$\begin{aligned}[\phi_+^{Aa}(\sigma),\psi_2(\sigma')]^* &= (-1)^A[\phi_+^{Aa}(\sigma),\psi_1(\sigma')] = \\ &= -2i(\gamma_-\acute{\theta}^A)^a(\sigma)\frac{(\Pi_A)^2(\sigma)}{\Pi_A^+(\sigma)}\delta(\sigma-\sigma') ,\end{aligned} \tag{39}$$

which shows that they are all first class. Thus, the rank of constraint matrix is 16. This is what one should predict, know the amount of gauge symmetry in the model. We now have a consistent Hamiltonian system, to which BRS quantization, say, c be applied, but this at the price of loss of manifest covaria

On the other hand, we can impose the light front gauge choices

$$\dot{X}^{+}(\sigma)=0\ ,\quad P^{+}(\sigma)=p^{+}\ ,$$
$$\theta_{-}^{Aa}(\sigma)=0\ , \tag{40}$$

together with the orthonormal gauge choice

$$(-g)^{1/2}g^{\alpha\beta}=\eta^{\alpha\beta}\ , \tag{41}$$

which leads us to the light front lagrangian (the light front hamiltonian is $p^{+}P^{-}$)

$$\mathcal{L}_{l.f.}=\dot{X}^{i}P^{i}+\dot{\bar{\theta}}^{A}_{a}P_{\theta}^{Aa}-p^{+}P^{-}=$$
$$=-\frac{1}{2}\eta^{\alpha\beta}\partial_{\alpha}X^{i}\partial_{\beta}X^{i}+ip^{+}\bar{\theta}\gamma_{+}\rho^{\alpha}\partial_{\alpha}\theta\ , \tag{42}$$

which, of course, is in accordance with ref. [2]. The non-zero Dirac brackets between the physical variables are

$$[X^{i}(\sigma),P^{j}(\sigma')]=\delta^{ij}\delta(\sigma-\sigma')\ , \tag{43}$$

$$\{\theta_{+}^{Aa}(\sigma),\theta_{+}^{Bb}(\sigma')\}=-i\delta^{AB}(\gamma_{-}C)^{ab}\frac{1}{4p^{+}}\delta(\sigma-\sigma')\ . \tag{44}$$

There is one possible objection to the argument above for the non-existence of a covariant gauge fixing procedure: We have demonstrated that the spinorial constraints contribute 16 second class constraints, and it remains to be investigated whether they can all be collected into one SO(1,9) spinor. Consider the constraint matrix (evaluated with the naive Poisson brackets)

$$A^{ab}(\sigma,\sigma')=\{\phi^{1a}(\sigma)+a\phi^{2a}(\sigma),\phi^{1b}(\sigma')+a\phi^{2b}(\sigma')\}=$$
$$=2i(\gamma_{\mu}C)^{ab}[(a^{2}+1)(P^{\mu}+S^{\mu})+(a^{2}-1)\acute{Z}^{\mu}](\sigma)\delta(\sigma-\sigma')\ . \tag{45}$$

Its inverse is

$$(A^{-1})_{ab}(\sigma,\sigma')=\frac{1}{2}i(C\gamma_{\mu})_{ab}\frac{(a^{2}+1)(P^{\mu}+S^{\mu})+(a^{2}-1)\acute{Z}^{\mu}}{[(a^{2}+1)(P+S)+(a^{2}-1)\acute{Z}]^{2}}(\sigma)\delta(\sigma-\sigma')$$
$$=\frac{1}{8a^{2}}i(C\gamma_{\mu})_{ab}\frac{(a^{2}+1)(P^{\mu}+S^{\mu})+(a^{2}-1)\acute{Z}^{\mu}}{(P+S)^{2}}(\sigma)\delta(\sigma-\sigma')\ . \tag{46}$$

The last quality is obtained using the constraint (24). Unfortunately, Eq. (46) is singular when $(P+S)^{2}=0$. We belive that this is inconsistent, and consequently the second class constraints cannot be covariantly eliminated. We do not know how to demonstrate the inconsistency, however, and therefore we proceed.

When $\phi_1+a\phi_2$ is taken to be second class, the remaining constraints (ψ_1, ψ_2 and $a\phi_1-\phi_2$) obey a closed Dirac bracket algebra among themselves. If we disregard the fact that the various Dirac brackets involve division by $(P+S)^2$, we would conclude that we have obtained a consistent Hamiltonian system without giving up manifest covariance. No gauge fixing has been performed at this point. If one sets a=1 and imposes the gauge choice $\theta_1-\theta_2=0$ to solve for the first class constraint $\phi_1-\phi_2=0$, one arrives at the model described by the action (17)! In this gauge $S^{\mu}=0$, and division by P^2 has been performed, which clear is inconsistent for massless strings. In general, the superchar

Q_1+aQ_2 contains enough degrees of freedom to create a massive supermultiplet. What one can not do is to fix the gauge in such a way that one obtains the light front description of the superstrings of ref. [2]. Therefore, the covariant Hamiltonian system does not describe these superstrings.

In conclusion then, there are two possibilities. Either, the second class constraints cannot be covariantly eliminated, and consequently the action (23) admits no covariant gauge, or else, division by $(P+S)^2$ is allowed (we stress that we do not believe this), but in this case the action (23) gives rise to two inequivalent Hamiltonian systems, of which only the noncovariant one describes the superstrings of ref. [2].

3. Conclusions

We have seen how the constraint structure of the covariant N=2 superstring theory in 10-dimensional space-time allows us to choose gauge conditions that make it reduce to the light-front theory earlier known. At the same time we have shown, that the structure is such, that it does not allow covariant gauge fixing (in any conventional sense; at least not for the superstrings of ref. [2]). The same statement holds true for the massless supersymmetric point-particle. This phenomenon is a serious obstacle to covariant quantization of the model; no conventional method is applicable. A result like this should however not be taken merely as negative. It has been realized in recent years that fully covariant formalisms for certain co plex models, involving extended supersymmetry and higher spin may become quite unwieldy and may not even exist. The problem noted have shed some light on the difficulties involved. On th other hand, the non-covariant light front method has been suc cessfully applied to superstrings. This suggests that the fur ther development of non-covariant methods may be important.

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Appendix on conventions and notation

The Regge slope parameter is taken to be

$$\alpha' = \frac{1}{2\pi} , \tag{A1}$$

space-like metric is used throughout,

$$\eta^{\mu\nu} = \mathrm{diag}(-1, 1, \dots 1) \tag{A2}$$

$$\eta^{\alpha\beta} = \mathrm{diag}(-1, 1) . \tag{A3}$$

Derivation with respect to the parameters σ and τ is denoted

$$\dot{A} = \frac{\partial}{\partial\tau} A \qquad\qquad A' = \frac{\partial}{\partial\sigma} A . \tag{A4}$$

For Majorana spinors in the 10-dimensional space,

$$\bar{\lambda}_a = -\lambda^b C_{ba} = -\lambda^b \gamma^0_{ba} \tag{A5}$$

and in the 2-dimensional parameter space,

$$\bar{\lambda}^c = -\lambda^d \rho^0_{dc} \tag{A6}$$

where γ^μ and ρ^α are Dirac matrices:

$$\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu} \tag{A7}$$

$$\{\rho^\alpha, \rho^\beta\} = -2\eta^{\alpha\beta} \tag{A8}$$

Plus and minus components of a vector are defined by

$$A^{\pm} = \frac{1}{\sqrt{2}} (A^0 \pm A^9) \tag{A9}$$

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