SOME PROBLEMS TO SOLVE

**A**. By inspection of Maxwell's equations show that the dimension of $c$ is that of length/time. Starting from the equations in SI units, rescale $\mathbf{E}$ and $\mathbf{B}$ with dimensionful constants so they get the same dimension.

**B**. Assuming that Stokes' theorem holds in three dimensions prove that Gauss' theorem holds in two dimensions. (Hint: In two dimension there is a unique direction normal to any given vector.)

**C**. Prove entry 5 in Jackson's list of properties of the delta function (p. 26).

**D**. An analytic function is a complex valued function on a two dimensional plane such that it depends only on the particular combination $z = x + iy$, $\Phi(x, y) = \Phi(x + iy) = \Phi(z)$. Prove that every such function is a solution of the two dimensional Laplace equation.

**E**. Use the chain rule for derivatives to express the two dimensional Laplace operator in polar coordinates ($x = r\cos\phi$, $y = r\sin\phi$).

**F**. The Poisson equation $\nabla^2\Phi = 4\pi G\rho_{\mathrm{m}}$, where $\rho_{\mathrm{m}}$ is the mass density, occurs in Newton's theory of gravity. Show that the potential

$$\Phi(x, y, z) = -\frac{GM}{\sqrt{x^2 + y^2 + (|z| + a)^2}} \tag{1}$$

comes from an infinitely thin disk of matter (that is, roughly, a spiral galaxy).

**G**. Create an electric field inside an spherical cavity. You can choose the boundary conditions freely. Can you choose them so that a test charge has a stable equilibrium position at the centre of the cavity? Anywhere in the interior of the cavity?

**Ga**. In his eq. (1.58) Jackson includes an integral. He says it is easy to show that it evaluates to $4\pi$. Verify this.

**H**. A small charge $q$ sits outside a spherical conductor of large total charge $Q$ and radius $R$. The charges have the same sign. At (roughly) what distance $d$ from the centre will the charge feel no force?

**Ha**. Two conducting planes are placed at $x = 0$ and at $y = 0$. The potential on the planes is kept at zero. Place a single charge somewhere in the positive quadrant of the $(x, y)$-plane. Use the method of images to calculate the electric potential in that quadrant.

**Hb**. Consider two conducting spheres, radii $a$ and $b$, distance between centres $c$, the first sphere at $\Phi = 0$, the other at $\Phi = V$. Place imaginary charges inside the spheres until the boundary conditions hold, in a stepwise procedure so that the first imaginary charge ensures that $\Phi = V$ on the second sphere, the second charge is inside the first sphere and compensates for the first charge so that $\Phi = 0$ on the second sphere. Go on in this way. It is enough to do the first few steps, but after each step you should use a computer to plot the potential at the surface of the first sphere, for suitable values of the parameters. This will give you a feeling for how quickly the procedure converges.

**Hc**. Derive Jackson's eq. (2.13) from his eq. (2.12), in full detail.

**I**. Compute the first five Legendre polynomials by applying Gram-Schmidt orthogonalization to the polynomials 1, $x$, $x^2$, $x^3$, $x^4$. Also show that you recover the first five Legendre polynomials by expanding the generating function $g(t,x) = (1 - 2xt + t^2)^{-1/2}$ to fourth order in $t$.

**J**. Show that the Legendre polynomial $P_l(x)$ has exactly $l$ zeroes in the interval.

**K**. Find the electric field outside a spherical conductor placed in a constant electric field in two ways: using the method of images, and using an expansion in spherical harmonics.

**L**. In Problem 2.16 Jackson gives an explicit solution for the electric potential inside a square, with a constant charge density there. Is this formula practically useful for plotting the potential? If it is, plot it!

**M**. Compute the first non-vanishing multipole moments for i) two charges $q$ at $(\pm a, 0, 0)$, charge $-2q$ at $(0, 0, b)$ ii) four charges $q$ at $(\pm a, \pm a, 0)$, two charges $-2q$ at $(0, 0, \pm b)$. Check your results.

**N** You have a supply of point charges $\pm q$. Place such point charges at the corners of a regular octahedron. You can make the monopole moment vanish. Can you make the dipole moment vanish? The quadrupole moment? Repeat the same exercise for a regular cube.

**O**. For a dipole field, locate those points in space where the field points in a direction orthogonal to the dipole vector.

**P**. A rectangular prism has a square base of area $A$ and a very small height $h$. Make a cavity of this form inside a dielectric. What is the electric field inside the cavity if the constant $\mathbf{D}$ in the dielectric is parallel to the height of the cavity? If $\mathbf{D}$ is orthogonal to the height?

**Q**. Model an atom as a homogeneous sphere of negative charge, and a point-like nucleus. Assuming a reasonable size, what electric field strength would be needed to shift the nucleus out to one tenth of the radius?

**R**. For pure nitrogen the following data were obtained (in 1934):

| | | | | |
|---|---|---|---|---|
| Temperature (°C): | 23.8 | 23.8 | 23.8 | 23.8 |
| Pressure (atm) | 1.02 | 57.5 | 221.6 | 1011.6 |
| Density (kg/m$^3$) | 1.18 | 66.04 | 236.1 | 578.0 |
| Dielectric constant | 1.00052 | 1.03109 | 1.11413 | 1.29633 |

Check how well this agrees with the Clausius-Mossotti equation.

**S**. Define a vector potential on a region of space strictly outside the $z$-axis, such that $\mathbf{A}(\mathbf{x})$ is independent of $z$, gives a vanishing magnetic field, and cannot be gauge transformed to zero. What is its physical interpretation?

**T**. Solve the Lorentz equation for a charged particle in a homogeneous magnetic field. Based on this solution, would you expect a plasma to be diamagnetic? Paramagnetic?

**U**. The energy of a magnetic dipole placed in a static magnetic field is $U = -\mathbf{m}\cdot\mathbf{B}$. There are three cases: permanent magnets ($\mathbf{m}$ constant), paramagnets and diamagnets ($\mathbf{m} = k\mathbf{B}$ with $k > 0$ for paramagnets and $k < 0$ for diamagnets). For each case investigate whether you can choose $\mathbf{B}(\mathbf{x})$ such that a stable equilibrium position for the dipole exists.

**V**. Consider the discontinuous function $\Psi(\mathbf{x}, t) = \Theta(f(\mathbf{x}, t))\psi(\mathbf{x}, t)$, where $\Theta$ is the step function (equals 1 if $f > 0$, 0 if $f < 0$). Find the conditions on $f$ and $\psi$ that ensure that we have a solution of the wave equation.

**W**. Using the retarded Green function solve the inhomogeneous wave equation

$$\Box\psi = \Theta(t)F(t)\delta^{(3)}(x) . \tag{2}$$

("Alice sits at the origin and starts talking at $t = 0$.") Does the solution make sense?

**X**. In a $d$-dimensional space the Green function for the Laplace equation is (up to a constant) $G(r) = 1/r^{d-2}$. Prove that $\delta(r - t)/r^{d-2}$ is a Green function for the wave equation if and only if $d = 3$.

**Y**. Draw a map of spacetime in which the $x$-axis and the $t$-axis appear orthogonal. Also draw a lightcone from the origin. Now perform a Lorentz boost.

Draw the $x'$-axis and the $t'$-axis on the map. How do they relate to the light cone?

**Z**. Consider a scalar field having the form

$$\phi(x) = \frac{1}{r^2} , \qquad r^2 = x^2 + y^2 + z^2 . \tag{3}$$

Perform a Lorentz boost in the $t - x$-plane, and express the new function $\phi'$ that you obtain in this way as a function of the coordinates $(t, x, y, z)$. What does the new function look like?

**Å**. Consider the electromagnetic field from a point charge at rest at the origin,

$$E_i(x) = \frac{1}{r^3} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad B_i(x) = 0 . \tag{4}$$

Perform a Lorentz boost in the $t - x$-plane. Compute the electromagnetic field you obtain, and express it as a function of the coordinates $(t, x, y, z)$.

**Ä**. Repeat exercise Å, but now for the electromagnetic field

$$E_2 = \cos(t - x) \qquad B_3 = \cos(t - x) , \tag{5}$$

all other components vanishing.

**Äa**. Prove that Maxwell's equations are not invariant under the Galilei transformation $t \to t' = t,\ \mathbf{x} \to \mathbf{x}' = \mathbf{x} + \mathbf{v}t$.

**Ö**. Use the expression for the energy density of the electromagnetic field to argue that the electric field in a wave sent out in all directions by a localized source falls off like $1/r$.