INTRODUCTION TO JACKSON

Fall, 2019.

Since Jackson's introduction is meant to be read only once you have understood the course, I provide one which is meant to be read first. Some problems, and rather detailed notes on Chapter 1, are included.

Maxwell's equations for the electromagnetic field are

$$\nabla \cdot \mathbf{B} = 0 \qquad \text{Gilbert's law} \tag{1}$$

$$\nabla \times \mathbf{E} + \partial_t \mathbf{B} = 0 \qquad \text{Faraday's law} \tag{2}$$

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho \qquad \text{Gauss' law} \tag{3}$$

$$\nabla \times \mathbf{B} - \frac{1}{c^2}\partial_t \mathbf{E} = \mu_0 \mathbf{j} \qquad \text{Ampère-Maxwell's law ,} \tag{4}$$

where $c^2 = 1/\epsilon_0\mu_0$. I use the notation

$$\partial_t = \frac{\partial}{\partial t} \ . \tag{5}$$

The charge density $\rho = \rho(\mathbf{x}, t)$ and the current density $\mathbf{j} = \mathbf{j}(\mathbf{x}, t)$ give a complete account of all charges, and once this has been provided the theory is complete.

A classical charged particle moving in an electromagnetic field is subject to Lorentz' force law

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \ , \tag{6}$$

where q is the charge of the particle. Originally the electric and magnetic fields were regarded as a way to encode the force acting on charged particles, but the equations show that fields live a life of their own.

1.1 UNITS

The constants ϵ_0, μ_0 depend on the unit system chosen. To make this choice one begins by making contact with the mechanical units through Coulomb's law: using Maxwell's equations one can show (and we will) that the force between two static charges at distance r metres from each other is

$$F = \frac{1}{4\pi\epsilon_0}\frac{qq'}{r^2} \ . \tag{7}$$

Therefore the dimension of q^2/ϵ_0 is that of [kg×metre^3/second^2], but the dimension and numerical value of ϵ_0 will depend on the unit adopted for charge. There is a similar story for μ_0. On the other hand the dimension of the constant c must be [metre/second], otherwise the left hand sides of eqs. (2) and (4) would be mutually inconsistent. Its numerical value in these units—close to $3 \cdot 10^8$—can be fixed by experiments.

In the SI system, which we will use, one makes the arbitrary choices

$$\epsilon_0 = \frac{10^7}{4\pi c^2} \ , \qquad \mu_0 = 4\pi \cdot 10^{-7} \ . \tag{8}$$

This comes from having adopted the Ampère as a basic unit for current, in terms of the force acting between two parallel wires. The factor of 4π is there essentially because the unit sphere has area 4π. Trying to suppress this in the equations will make it pop up somewhere else. To reach the more intelligent Gaussian unit system one sets $F = qq'/r^2$, insists that the electric and magnetic fields have the same dimension, and makes sure that the only constants that appear in the equations are c and 4π.

Jackson switches to Gaussian units later in the book.[1] He has a nice discussion of units on pp. 775-784. Read it carefully!

1.2 VECTOR ANALYSIS

A boldface letter denotes a vector, say

$$\mathbf{E} = \mathbf{e}_x E_x + \mathbf{e}_y E_y + \mathbf{e}_z E_z \ . \tag{9}$$

If the basis vectors are understood one can think of a vector as an array of numbers,

$$\mathbf{E} = \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \ . \tag{10}$$

This is sloppy but useful. Finally, $\mathbf{E}(\mathbf{x})$ is a *vector field*, a vector defined at each point of space, and can be partially visualized by its field lines, which go through the points in the direction of the vector defined there.

The differential operator ∇ is called "nabla", Hebrew for the musical instrument known in English as "psalter"—because this is what its symbol looked like to Maxwell, who knew his Bible. When it acts on a function it gives the *gradient* of the function, which is the vector field

[1] Everyday quantities come out as everyday numbers in Gaussian units. In the lecture room the geomagnetic field is about half a Gauss, and about 50 microteslas in SI units.

$$\nabla f = \mathbf{e}_x \partial_x f + \mathbf{e}_y \partial_y f + \mathbf{e}_z \partial_z f \ . \tag{11}$$

Nabla operates on a vector field $\mathbf{E}$ in two different ways. The *divergence* of $\mathbf{E}$ is the function

$$\nabla \cdot \mathbf{E} = \partial_x E_x + \partial_y E_y + \partial_z E_z \ . \tag{12}$$

(I am assuming a Cartesian coordinate system.) The *rotation* or *curl* of $\mathbf{E}$ is the vector field

$$\nabla \times \mathbf{E} = \begin{pmatrix} \partial_y E_z - \partial_z E_y \\ \partial_z E_x - \partial_x E_z \\ \partial_x E_y - \partial_y E_z \end{pmatrix} \ . \tag{13}$$

(Note the cyclic pattern. Note also that the notation has been streamlined to fit a three dimensional space.) Applying nabla repeatedly we obtain the *Laplace operator*

$$\nabla^2 = \nabla \cdot \nabla = \partial_x^2 + \partial_y^2 + \partial_z^2 \ . \tag{14}$$

It turns functions into functions and vectors into vectors. In a purely automatic manner one proves the identities

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} \tag{15}$$

$$\nabla \cdot (\nabla \times \mathbf{E}) = 0 \ , \qquad \nabla \times (\nabla f) = 0 \ . \tag{16}$$

The interpretation of gradient, divergence, and curl becomes clear once one has studied the three Wonderful Theorems of vector analysis.

The first of these theorems observes that the integral of a gradient of a function along a curve C depends only on the end points $\mathbf{x}_1$ and $\mathbf{x}_2$ of the curve, not on where the curve goes in between. So

$$\int_C \nabla f \cdot \mathbf{dl} = f(\mathbf{x}_2) - f(\mathbf{x}_1) \ . \tag{17}$$

(You are supposed to remember how curve and surface integrals are defined, in particular that $\mathbf{dl}$ is a vector directed along the tangent vector of the curve.) An equivalent way of stating this result is to say that the integral of a gradient along a closed curve vanishes. (Why is it equivalent?) The wonderful thing is that the converse holds: If the integral along *every* closed curve vanishes the vector field is necessarily a gradient,

$$\oint_C \mathbf{A} \cdot \mathbf{dl} = 0 \qquad \Leftrightarrow \qquad \mathbf{A} = \nabla f \ , \tag{18}$$

or more completely in words: *The integral of a vector field* $\mathbf{A}$ *vanishes around every closed loop if and only if that vector field is a gradient of a function.* This function is determined up to an arbitrary constant. In the proof one picks an arbitrary point $\mathbf{x}_1$ and defines the function f by means of

$$f(\mathbf{x}) = \int_C^{\mathbf{x}} \mathbf{A} \cdot dl \ , \tag{19}$$

where the curve starts at $\mathbf{x}_1$ and ends at $\mathbf{x}$. The definition does not depend on the choice of the curve because $\mathbf{A}$ is such that its integral along any closed curve vanishes, and the dependence on $\mathbf{x}_1$ is only through an overall constant.

There is some fine print: The functions have to be differentiable and the curve has to be smooth enough so that the integrals exist. It is also assumed that all closed curves are *contractible*—each curve has to bound a surface lying entirely inside the space. If space is like the surface of a cylinder then curves winding around the cylinder cannot be shrunk to a point without leaving the cylinder, and such closed curves are said to be non-contractible. This is an interesting caveat, and we will actually come across it.

The next theorem is

Stokes' theorem: *For every surface* S *with boundary* C *and for every vector field* $\mathbf{A}$ *there holds*

$$\int_S (\nabla \times \mathbf{A}) \cdot \mathbf{da} = \oint_C \mathbf{A} \cdot \mathbf{dl} \ . \tag{20}$$

The surface area element is $\mathbf{da} = \mathbf{n} dS$, where $\mathbf{n}$ is the unit normal vector of the surface. There is a sign which is determined by the right hand thumb rule. A closed surface is one without any boundary, and it follows that the integral of the curl of a vector field over any closed surface S vanishes. Again the converse holds: If the integral over *every* closed surface vanishes the vector field is necessarily a curl,

$$\oint_S \mathbf{B} \cdot \mathbf{da} = 0 \qquad \Leftrightarrow \qquad \mathbf{B} = \nabla \times \mathbf{A} \ . \tag{21}$$

The vector field $\mathbf{A}$ is determined up to a gradient of an arbitrary function. And again there is some fine print which we ignore.

You may think that these theorems are useless in practice: for a given vector field we will never be able to test if the integral vanishes around every closed loop, or over every closed surface. But sometimes we can. Stokes' theorem says that the integral over every closed loop will vanish if the vector field has vanishing curl. Then the converse of the first theorem ticks in to show (modulo the fine print) that

$$\nabla \times \mathbf{A} = 0 \qquad \Leftrightarrow \qquad \mathbf{A} = \nabla f \ . \tag{22}$$

The third and final theorem is

Gauss' theorem: *For every volume V with boundary S and for every vector field* $\mathbf{B}$ *there holds*

$$\int_V \nabla \cdot \mathbf{B} \mathrm{d}V = \oint_S \mathbf{B} \cdot \mathbf{da} \ . \tag{23}$$

The right hand side is known as the *flux* of $\mathbf{B}$ through S. If a vector field $\mathbf{B}$ has vanishing divergence its flux through every closed surface that bounds a volume vanishes, and Stokes' theorem then ensures that there exists a vector field $\mathbf{A}$ such that

$$\nabla \cdot \mathbf{B} = 0 \qquad \Leftrightarrow \qquad \mathbf{B} = \nabla \times \mathbf{A} \ . \tag{24}$$

Using Stokes' and Gauss' theorems one can formulate Maxwell's equations in words, and recapture Faraday's intuitions about the electromagnetic field. ("The total flux of a magnetic field through a closed surface vanishes", "the integral of the electric field along a closed loop equals the time derivative of the flux of a magnetic field through any surface spanned by the loop", and so on.)

1.3 IMPORTANT APPROXIMATIONS

Our description of the right hand side of Maxwell's equations was a bit vague. One statement can be made right away. If we apply a time derivative to the charge density we can use the equations to see that

$$\partial_t \rho = \nabla \cdot \partial_t \mathbf{E} = \nabla \cdot (c^2 \nabla \times \mathbf{B} - \mu_0 \mathbf{j}) = -\mu_0 \nabla \cdot \mathbf{j} \ . \tag{25}$$

Thus electric charge is a conserved quantity; using Gauss' theorem we can change the differential equation to the verbal statement that the total change of the charge within a given volume V is equal to the *flux* of the current through its boundary.

A complete physical theory should provide a more detailed description. We can imagine that the charges occur in the form of point particles subject to Lorentz' force law, and that ρ and $\mathbf{j}$ are defined as suitable spatial averages over these charges. But by then the equations are extremely difficult to solve, since the electric and magnetic fields affect the right hand sides in such a way that the equations become non-linear. There are two approximations that simply ignore this difficulty:

1: External charges: ρ and $\mathbf{j}$ are known. We solve for $\mathbf{E}$ and $\mathbf{B}$.

2: External fields: $\mathbf{E}$ and $\mathbf{B}$ are known. We solve for the motion of the charges.

It frequently happens that only a part of the charge distribution can be controlled externally, while another part is distributed throughout some piece of matter in some manner not known in detail. The latter is then moved to the left hand side of the equations by replacing $\mathbf{E}$ and $\mathbf{B}$ with the fields $\mathbf{D} = \mathbf{D}[\mathbf{E}, \mathbf{B}]$ and $\mathbf{H} = \mathbf{H}[\mathbf{E}, \mathbf{B}]$ in eqs. (2) and (4). The microscopic properties of matter are summarized by these relations. The dependence of $\mathbf{D}$ on the microscopic fields $\mathbf{E}$ and $\mathbf{B}$ is denoted by square rather than round brackets because the function may not be single valued. There may be a dependence on the history of the system. There is certainly a dependence on the frequency (time dependence) of the applied fields. The resulting theory is called electrodynamics in a medium. We will arrive at it in due course. If there is no medium we set $\mathbf{D} = \epsilon_0 \mathbf{E}$ and $\mathbf{H} = \mathbf{B}/\mu_0$.

A dramatic simplification of Maxwell's equations occurs in *stationary* situations, where nothing depends on the time t. Then the equations split into two independent sets, the *electrostatic equations*

$$\begin{cases} \nabla \times \mathbf{E} = 0 \\ \nabla \cdot \mathbf{D} = \rho \end{cases} \tag{26}$$

and the *magnetostatic equations*

$$\begin{cases} \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{H} = \mathbf{j} \quad . \end{cases} \tag{27}$$

Experimentally these equations were established first. Electricity and magnetism seemed to be two disconnected phenomena.

1.4 OPTICS

If we set $\rho = 0$ and $\mathbf{j} = 0$ and take a quick look at the equations, something remarkable emerges: Applying a curl to Faraday's law, and recalling that all the right hand sides of Maxwell's equations now vanish, we see that

$$0 = \nabla \times (\nabla \times \mathbf{E} + \partial_t \mathbf{B}) = \nabla \nabla \cdot \mathbf{E} - \nabla^2 \mathbf{E} + \partial_t \nabla \times \mathbf{B} = -\nabla^2 \mathbf{E} + \frac{1}{c^2} \partial_t^2 \mathbf{E} = 0 \; . \tag{28}$$

The components of $\mathbf{E}$ (and by a similar calculation the components of $\mathbf{B}$) obey the *wave equation*, describing the propagation of waves with velocity $c \approx 3 \cdot 10^8$ m/s. The numerical coincidence between the constant c and the velocity of light was noted by Kirchhoff. Maxwell analyzed things further, and then made the comment:

We can scarcely avoid the inference that light consists in the transverse undulations of some medium which is the cause of electric and magnetic phenomena.

Maxwell's equations bring all electrical, magnetical, and optical phenomena under the same hat.

1.5 WHY THIS SUBJECT?

Classical electrodynamics is not an active research area. On the other hand Maxwell's equations, together with a proper (perhaps quantum mechanical) description of charged matter, covers basically all of physics from the scale of atoms to the scale of bacteria, inclusive. At smaller scales the nuclear forces begin to be important. At larger scales gravity kicks in. But if you want to explore any phenomenon in between electrodynamics is the only force that counts. Frictional forces, van der Waals forces, light, and more, are manifestations of electromagnetism. As a service science, electrodynamics stands supreme.

Moreover you have to base any understanding of the other forces on your understanding of electrodynamics, which is the simplest of the fundamental force fields. Among the latter electrodynamics is unique in that there is a large range of observed phenomena for which the classical version is relevant, and a large range for which the quantum version is needed. With the nuclear forces all is quantum, with gravity all is classical.

Fundamental research is often defended by quoting the famous answer to a politician who asked what use one can make of it: "What is the use of a new-born baby?". The story is often told about Gladstone and Faraday (and if you cannot recall who Gladstone was, never mind). Anyway the story concerns research into the electromagnetic field—and the range of practical applications that this research eventually turned out to have is simply fantastic. Our technological civilization rests on it.

PROBLEMS

1. Prove eqs. (15) and (16).

2. For arbitrary functions f and g and arbitrary volumes V with boundary S, prove that

$$\int_V (f\nabla^2 g + \nabla f \cdot \nabla g) dV = \int_S f \nabla g \cdot \mathbf{da} \ . \tag{29}$$

3. Also prove that

$$\int_V (f\nabla^2 g - g\nabla^2 f) dV = \int_S (f\nabla g - g\nabla f) \cdot \mathbf{da} \ . \tag{30}$$

4. Compute the flux of the vector field $\mathbf{v}$ through the unit sphere, where

$$\mathbf{v} = 3xy\mathbf{e}_x + xz^2\mathbf{e}_y + y^3\mathbf{e}_z \ . \tag{31}$$

5. Let $f = x^2 y e^z$ and let C be the curve $\sigma(t) = (\sin t, \cos t, t)$, $0 \leq t \leq 2\pi$. Compute $\int_C \nabla f \cdot \mathbf{dl}$.

6. By inspection of Maxwell's equations show that the dimension of c is that of length/time. Starting from the equations in SI units, rescale $\mathbf{E}$ and $\mathbf{B}$ with dimensionful constants so they get the same dimension.

2 — JACKSON, CHAPTER 1

In electrostatics the electric field really is just a device encoding the force that acts on a static test charge. This force is due to the presence of some other static charges. For one charge placed at the origin there is a radial electric field

$$E_r = \frac{q}{4\pi\epsilon_0}\frac{1}{r^2} \ . \tag{32}$$

If there are several charges, at positions $\mathbf{x}_i$, one obtains the total electric field at position $\mathbf{x}$ by just adding the electric fields due to each individual charge,

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\sum_i q_i \frac{\mathbf{x}-\mathbf{x}_i}{|\mathbf{x}-\mathbf{x}_i|^3} \ . \tag{33}$$

This is called the Principle of Superposition. For a continuous charge distribution the sum becomes an integral,

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int d^3x' \ \rho(\mathbf{x}')\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} \ . \tag{34}$$

These formulas are well supported by experiments. They lead to two questions:

Q1: Are they equivalent to the partial differential equations

$$\begin{cases} \nabla \times \mathbf{E} = 0 \\ \\ \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho \ . \end{cases} \tag{35}$$

Q2: Assuming they are, what is the use of these differential equations?

The second question is similar to the question why Newton's description of the orbit of Mars is better than that of Kepler. One interesting answer is that we will frequently do electrostatics in situations where we do not know the charge density ρ. What we know is, say, that there are some charges that can move freely on a conductor. When we place an electric charge outside the conductor these charges will move, and eventually reach some equilibrium distribution ρ. We are asking for the electric field produced by that equilibrium distribution together with the extra charge, but we do not know ρ in advance. It turns out that the differential equations can handle this, but we save this issue for later.

The first question comes in two parts. We have to prove that the known solution solves the equations. But we also have to show that the solution is unique. We do not want spurious unphysical solutions to appear.

The first observation is that because the electric field has vanishing curl it is the gradient of some function $\Phi = \Phi(\mathbf{x})$, known as the *electric potential*. We rewrite the equations as

$$\begin{cases} \mathbf{E} = -\nabla\Phi \\ \\ \nabla^2\Phi = -\frac{1}{\epsilon_0}\rho \ . \end{cases} \tag{36}$$

The equation for Φ is known as *Poisson's equation*. It is the basic equation in Newtonian gravitation too, with the interesting difference that there the right hand side is assumed to be never negative.

The second observation is that the equations are linear in the unknown $\mathbf{E}$, or Φ. Therefore solutions can be added, the Principle of Superposition holds, and it is enough to show equivalence for the case of a single charge, conveniently placed at the origin. We know what $\mathbf{E}$ we want, and we observe that

$$-\nabla\left(\frac{1}{r}\right) = \frac{1}{r^2}\nabla\sqrt{x^2+y^2+z^2} = \frac{1}{r^3}(x\mathbf{e}_x + y\mathbf{e}y + z\mathbf{e}_z) = \frac{\mathbf{x}}{r^3} \ . \tag{37}$$

So we have found the electric potential for the electrostatic field from a point charge. Thus what we need to show is that

$$\Phi = \frac{q}{4\pi\epsilon_0}\frac{1}{r} \qquad \Leftrightarrow \qquad \nabla^2\Phi = -\frac{1}{\epsilon_0}\rho \ , \tag{38}$$

where ρ is the charge distribution describing a single charge q placed at the origin. It vanishes everywhere except at the origin itself, where it is presumably infinite.

It is easy to see that

$$r \neq 0 \qquad \Rightarrow \qquad \nabla^2\left(\frac{1}{r}\right) = 0 \ . \tag{39}$$

What I mean by "easy to see" is that it is easy to check. Using pen and paper we apply the derivatives to the function, collect terms, and find that they cancel. The terms will have some power of r in the denominator, which means that we know nothing about the behaviour when $r = 0$. But then ρ is "infinite" at the origin, and this is not a well defined concept. It means that we have to do some work in order to fully define ρ.

What we do know about ρ is that

$$\begin{cases} r \neq 0 \quad \Rightarrow \quad \rho(\mathbf{x}) = 0 \\ \\ \int_V \rho \ \mathrm{d}V = q \ , \end{cases} \tag{40}$$

where V is any volume containing the origin. So what we have to show is that $-\epsilon_0\nabla^2\Phi$ has the same properties, for the potential Φ we have chosen. But this is easy. We can safely assume that V is a round ball centred at the origin, since the integrand vanishes elsewhere. Then we apply Gauss' theorem to convert the volume integral to an integral over the surface of the ball:

$$\int_{\odot} \left(-\epsilon_0\nabla^2\Phi\right) \mathrm{d}V = -\frac{q}{4\pi}\int_{\odot} \nabla^2\left(\frac{1}{r}\right) \mathrm{d}V = \tag{41}$$

$$= -\frac{q}{4\pi}\oint_S \nabla\left(\frac{1}{r}\right)\cdot\mathbf{da} = \frac{q}{4\pi}\int_0^{\pi} d\theta\sin\theta\int_0^{2\pi} d\phi\frac{r^2}{r^2} = q\ .$$

This does it. The expression $(-\epsilon_0\nabla^2\Phi)$ has exactly the same properties as does the charge density ρ, hence they are equal.

2.2 DELTA FUNCTIONS AND THE FUNDAMENTAL SOLUTION

At this point it is convenient to introduce *delta functions*. A delta function, in one dimension, is defined as an object $\delta(x-a)$ such that

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a)\ , \tag{42}$$

for all functions $f(x)$ subject to some restrictions. A delta function is in itself not a function. It is an object that always occurs inside an integral. Nor does "the delta function" exist in isolation. A delta function is always defined relative to some particular class of functions $f(x)$ that one allows in the integral.

It is assumed that an integral in which a delta function occurs can be subject to all the usual manipulations such as changes of variables,

$$\int_{-\infty}^{\infty} f(x)\delta(ax)dx = \int_{-\infty}^{\infty} f(x/a)\delta(x)\frac{dx}{|a|} = \frac{1}{|a|}f(0)\ . \tag{43}$$

This is written symbolically as $\delta(ax) = \delta(x)/|a|$. Partial integration can be done too,

$$\frac{df}{dx}(0) = \int_{-\infty}^{\infty}\frac{df}{dx}\delta(x)dx = -\int_{-\infty}^{\infty} f(x)\frac{d}{dx}\delta(x)dx\ . \tag{44}$$

Reading this backwards gives a meaning to the derivative of a delta function. For our purposes the properties of the delta function are given by Jackson on p. 26.

What we did was to set the charge density of a point charge to

$$\rho(\mathbf{x}) = q\delta(\mathbf{x}) = q\delta(x)\delta(y)\delta(z) \ . \tag{45}$$

The whole notion of a charge concentrated to a point is of course a mathematical fiction, in classical and quantum electrodynamics alike. Classical electrodynamics really concerns charge densities that are well behaved continuous functions. This results in a well defined theory—even if it is based on a mathematical idealization that ignores the discrete nature of charges in Nature.

We then proceeded to show that the function

$$G(\mathbf{x}) = \frac{1}{r} \tag{46}$$

obeys

$$\nabla^2 G(\mathbf{x}) = -4\pi\delta(\mathbf{x}) \ . \tag{47}$$

This is called the *fundamental solution* of Poisson's equation. If we shift the origin it is usually called a *Green function*;

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{|\mathbf{x} - \mathbf{x}'|} \qquad \Rightarrow \qquad \nabla^2 G(\mathbf{x}, \mathbf{x}') = -4\pi\delta(\mathbf{x} - \mathbf{x}') \ . \tag{48}$$

This is now a function of two points in space, but like a delta function it is an object that is really supposed to sit inside an integrand. The point is that with a Green function in hand we can solve Poisson's equation for an arbitrary charge density ρ. The solution is

$$\Phi(\mathbf{x}) = \int d^3x' \frac{1}{4\pi} G(\mathbf{x}, \mathbf{x}') \frac{\rho(\mathbf{x}')}{\epsilon_0} = \int d^3x' \frac{1}{4\pi\epsilon_0} \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \ . \tag{49}$$

This is an integral that we can do, given ρ. In this sense it is explicit. Applying the gradient operator to it we recover the electrostatic field (34). And if we apply the Laplace operator we get

$$\nabla^2 \Phi(\mathbf{x}) = \int d^3x' \frac{\nabla^2 G(\mathbf{x}, \mathbf{x}')\rho(\mathbf{x}')}{4\pi\epsilon_0} = -\int d^3x' \frac{\delta(\mathbf{x} - \mathbf{x}')\rho(\mathbf{x}')}{\epsilon_0} = -\frac{\rho(\mathbf{x})}{\epsilon_0} \ . \tag{50}$$

We have an answer to the first part of the first question. Concerning Green functions we will have more to say later.

2.3 UNIQUENESS OF THE SOLUTION

We are not done with our questions, since we must show that the differential equations do not introduce spurious unphysical solutions. To investigate this,

suppose that we have two different solutions for the same charge distribution. The linearity of the equation then admits the deduction

$$\begin{cases} \nabla^2\Phi_1 = -\frac{1}{\epsilon_0}\rho \\ \\ \nabla^2\Phi_2 = -\frac{1}{\epsilon_0}\rho \end{cases} \qquad \Rightarrow \qquad \nabla^2(\Phi_1 - \Phi_2) = 0 \ . \tag{51}$$

Thus what we need to prove uniqueness is that

$$\nabla^2\Phi = 0 \qquad \Rightarrow \qquad \Phi = 0 \ . \tag{52}$$

It will turn out that this is true provided we add the extra condition that $\Phi \to 0$ as $r \to \infty$. This assumption makes physical sense ("we can ignore any charges or electric fields sitting far away from the region of interest"). The fact that extra assumptions have to be brought in is typical of laws formulated in terms of differential equations. Newton's laws do not in themselves predict that the orbit of Mars is an ellipse. You have to add initial conditions saying that at this moment in time Mars is at this position and travels with this velocity. Similarly, extra conditions will be needed to get definite statements out of Poisson's equation. One has to analyze the partial differential equation in detail in order to see precisely what kind of conditions that will make the solution unique. In this way partial differential equations differ from ordinary differential equations. In the latter case there are some general theorems about existence and uniqueness of solutions that apply to all ODEs. There is nothing quite like that for PDEs.

To quote Fritz John:

> The constraints imposed by a partial differential equation on its solutions (like those imposed by the environment on a living organism) have an infinite variety of consequences, local and global, equalities and inequalities.

After this dark remark we turn to a study of the *Laplace equation*

$$\nabla^2\phi = 0 \ . \tag{53}$$

We are going to establish a very important property that will enable us to produce the required uniqueness proof with both hands in our pockets.

- The Laplace equation has the Mean Value Property that

$$\nabla^2\Phi = 0 \qquad \Leftrightarrow \qquad \Phi(\mathbf{x}) = \frac{\int_S \Phi dS}{\int_S dS} \ , \tag{54}$$

  where the integration is over any sphere S centred at $\mathbf{x}$.

At any point the value assumed by Φ equals the average of the values it assumes on a sphere surrounding the point, and this is true for every such sphere.

Suppose first that we have proved this. It follows that

- The Laplace equation obeys the Maximum Principle: if the Laplace equation holds within a bounded volume V then any local maximum or minimum of the function must occur on the boundary of the volume.

This is obvious: if a local maximum occurs at a point in the interior we could surround that point with a very small sphere. On that sphere the values assumed by the function are smaller than the maximum, hence the average over that sphere cannot be equal to the value at the point, which contradicts the Mean Value Property.

- If $\nabla^2\Phi = 0$ in a region and if $\Phi = 0$ on the boundary surrounding that region it must be true that $\Phi = 0$ throughout the region.

This is is true because the Maximum Principle says that the maximum and the minimum of the function are both equal to 0.

- The solution of the Laplace equation in a region is uniquely determined by the values of the function on the boundary surrounding the region.

This follows because the difference of two such solutions vanishes at the boundary, and then it vanishes everywhere.

The values of Φ on the boundary surrounding a region in which the Laplace equation is to be solved are known as *Dirichlet conditions*. Dirichlet conditions determine the solution uniquely.

The problem we want to solve is only a little different. We were imposing the Laplace equation over all space. The claim is that the solution $\Phi = 0$ is unique if the function goes to zero at infinity. To prove that the value of $\Phi(\mathbf{x})$ is smaller than any given ϵ, surround the point $\mathbf{x}$ with a very large sphere. If the sphere is large enough the values assumed by Φ on that sphere will be smaller than ϵ. So will the average over that sphere. By the Mean Value Property it follows that $\Phi(\mathbf{x}) < \epsilon$, and this is true for all ϵ, hence $\Phi = 0$.

It remains to prove the Mean Value Property. But this is easy. First of all it is trivial in one dimension, where vanishing second derivative means that the function is linear. When we average a function Φ over a sphere we obtain a function of the position $\mathbf{x}$ of the centre of the sphere and of its radius r, namely

$$M_\Phi(\mathbf{x}, r) = \frac{\int_S \Phi dS}{\int_S dS} = \frac{1}{4\pi}\int \Phi d\Omega \ , \qquad d\Omega = d\theta d\phi \sin\theta \ . \tag{55}$$

Factors of r cancel since the surface element of the sphere S contains a factor of r^2. To see how the dependence on $\mathbf{x}$ and r comes in we write this formula in full detail,

$$M_\Phi = \frac{1}{4\pi}\int_0^\pi \int_0^{2\pi} \Phi(x + r\cos\phi\sin\theta, y + r\sin\phi\sin\theta, z + r\cos\theta)d\Omega \ . \qquad (56)$$

If we make the sphere very small the average will approach the value at the centre arbitrarily closely,

$$\lim_{r\to 0} M_\Phi(\mathbf{x}, r) = \Phi(\mathbf{x}) \ . \qquad (57)$$

Now we take the derivative with respect to r, and afterwards we reinstate the factors of r to have everything written in proper vector form:

$$\partial_r M_\Phi(\mathbf{x}, r) = \frac{1}{4\pi}\int \partial_r \Phi d\Omega = \frac{1}{4\pi r^2}\int_S \nabla\Phi \cdot \mathbf{da} \ . \qquad (58)$$

Inside the integral the derivative with respect to r was traded for the gradient scalared with the unit normal vector. Now we apply Gauss' theorem to convert this into an integral over a ball and use the fact that Φ solves the Laplace equation:

$$\partial_r M_\Phi(\mathbf{x}, r) = \frac{1}{4\pi r^2}\int_\odot \nabla^2\Phi dV = 0 \ . \qquad (59)$$

Hence the average does not depend on the radius of the sphere—it equals the average over an arbitrarily small sphere, and hence it equals the value of Φ at the centre.

This concludes our proof that

$$\left(\begin{array}{c}\text{Maxwell's equations}\\ +\\ \text{vanishing } \mathbf{E} \text{ at infinity}\end{array}\right) \quad \Leftrightarrow \quad \text{(known electrostatic fields)} \ . \qquad (60)$$

Problems:

1. Assuming that Stokes' theorem holds in three dimensions prove that Gauss' theorem holds in two dimensions. (Hint: In two dimension there is a unique direction normal to any given vector.)

2. An analytic function is a complex valued function on a two dimensional plane such that it depends only on the particular combination $z = x + iy$, $\Phi(x, y) = \Phi(x + iy) = \Phi(z)$. Prove that every such function is a solution of the two dimensional Laplace equation.

3 — CHAPTER 1 CONCLUDED

There remain some things to say about the conditions under which the solution of the Laplace equation is unique. These conditions will always be set on a surface that surrounds the region of interest, or perhaps at infinity. It is easy to see why conditions on surfaces are important. Consider an *ideal conductor*, on which charges can move freely. In equilibrium there cannot be any electric field inside, otherwise the charges would start to move. To prevent this from happening the free charges will move to the surface of the conductor, and stay there. (We will see later why they do not leave the conductor.) Once equilibrium has set in the potential will be constant all over the conductor, including its boundary, which means that we need to solve a Dirichlet problem to find the electric field outside it.

This also means that an ideal conductor will tend to have a surface charge density, vanishing everywhere except in an infinitely thin shell at the surface. (In a real metal, say copper, the surface charge is concentrated to a surface layer that is a few Ångströms across.) And this naturally brings up the question: given that we know the electrostatic field from a point charge, what will it be for line and surface charges?

3.1 ELECTROSTATICS IN LOWER DIMENSIONS

As long as we are close to the line or surface where the charge is concentrated we can assume that the line is straight and the surface is a plane. The only relevant directions are normal to the line or surface. In effect then we are interested in how electrostatics behaves if space has only two or one dimensions. Given that we know the answer for three dimensions this is an interesting and easy exercise.

Gauss' theorem holds in all dimensions, so for a point charge q it remains true that

$$\oint_S \mathbf{E} \cdot \mathbf{da} = \frac{q}{\epsilon_0} \ , \tag{61}$$

where the integration is over a "sphere" surrounding the origin. If the dimension $d = 2$ this sphere is a circle, if $d = 1$ it consists of two points equidistant from the charge. Since the area of the sphere grows like r^{d-1} we conclude that the strength of the electric field falls like

$$E \sim \frac{1}{r^{d-1}} \ . \tag{62}$$

If $d = 2$ we get for the electric field and the potential from a point charge that

$$E \sim \frac{1}{r} \ , \qquad \Phi \sim \ln r \ , \qquad r = \sqrt{x^2 + y^2} \ . \tag{63}$$

This is singular at the origin, but less so than in three dimensions. The same behaviour will be observed close to a line charge in $d = 3$.

If $d = 1$, or if we are close to a surface charge in three dimensions, the electric field does not fall off with distance at all. It still points away from (or towards) the charge though, so it changes sign there. Indeed

$$E = \begin{cases} \frac{q}{2\epsilon_0} & \text{if } x > 0 \\ \\ -\frac{q}{2\epsilon_0} & \text{if } x < 0 \ . \end{cases} \tag{64}$$

This step function is the derivative of the potential, and we see that the potential has a kink at the position of the charge. The charge density is the derivative of a step function, hence it vanishes everywhere except at the origin, and there it is "infinite". It looks suspiciously like a delta function, and in fact it is. If we have a step function $\epsilon(x)$ which jumps from $-1/2$ to $+1/2$ at the origin we do a partial integration to see that

$$\begin{aligned} \int_{-\infty}^{\infty} f(x) \frac{d\epsilon(x)}{dx} \mathrm{d}x = \frac{1}{2}(f(\infty) - f(-\infty) - \int_{-\infty}^{\infty} \frac{df}{dx} \epsilon(x) = \\ = \frac{1}{2}(f(\infty) - f(-\infty)) - \frac{1}{2}\left(\int_0^{\infty} \frac{df}{dx} \mathrm{dx} - \int_{-\infty}^{0} \frac{df}{dx} \mathrm{d}x \right) = f(0) \ . \end{aligned} \tag{65}$$

This calculation proves that the derivative of the step function is a perfectly respectable one dimensional delta function, provided only that we restrict ourselves to functions $f(x)$ that have a finite limit as $x \to \pm\infty$.

However, we brought up electrostatics in $d = 1$ because we want to understand what can happen close to a surface charge density in $d = 3$, not because we want to study delta functions further. Translated to three dimensions our deliberations mean that the normal component of the electric field will be discontinuous, and that the jump is given by

$$(\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}}) \cdot \mathbf{n} = \frac{1}{\epsilon_0} \sigma \ , \tag{66}$$

where $\mathbf{n}$ is the unit normal vector of the surface and σ the surface charge density—which may vary as we move along the surface. This can be proved directly in three dimensions using Gauss' law, and a suitable choice of volume straddling the surface. See the illustration in Jackson's Introduction. A similar argument applied to the magnetic field shows that the normal component of $\mathbf{B}$ must be continuous, unconditionally.

In electrostatics the tangential component of the electric field is continuous across the boundary. Again this can be shown directly from Maxwell's equations. It is interesting to temporarily drop the restriction to time independent field here, and use the full Maxwell equations. Draw a closed loop in space again surrounding a thin strip S crossing the boundary. We have

$$\oint \mathbf{E} \cdot \mathbf{dl} = \int_S \nabla \times \mathbf{E} \cdot \mathbf{da} = -\int_S \partial_t \mathbf{B} \cdot \mathbf{da} \ . \tag{67}$$

The shape of the loop is chosen according to the illustration in Jackson's Fig. I.4. In particular its area goes to zero. The only contributions to the left hand side are coming from the tangential components of the electric field on both sides of the boundary. Because the area of the loop can be chosen arbitrarily small the right hand side will vanish as long as the time derivative of the magnetic field assumes finite values at the boundary. In electrostatics it follows that the tangential component must be continuous. But in electrodynamics we can have solutions, in charge free space, such that the electric and magnetic fields vanish on one side of a surface, and 'jump' to finite values on the other side. The 'jump' affects only the tangential components of $\mathbf{E}$ and $\mathbf{B}$.

3.2 ELECTROSTATIC ENERGY

Now for something different, which will add some 'reality' to the electric field. The work done on an electric point charge when we move it from A to B in a static electric field is easily computed. Lorentz' force law gives

$$W = -\int_A^B \mathbf{F} \cdot \mathbf{dl} = -q\int_A^B \mathbf{E} \cdot \mathbf{dl} = q\int_A^B \nabla\Phi \cdot \mathbf{dl} = q(\Phi_B - \Phi_A) \ . \tag{68}$$

This gives physical meaning to potential differences, but not to the actual value.

We can then ask how much energy one has to supply in order to create a configuration of several point charges by moving them in from infinity (far away). The answer becomes a sum over all pairs of charges,

$$W = \frac{1}{4\pi\epsilon_0} \sum_{\text{pairs}} \frac{q_i q_j}{|\mathbf{x}_i - \mathbf{x}_j|} = \frac{1}{8\pi\epsilon_0} \sum_{i \neq j} \frac{q_i q_j}{|\mathbf{x}_i - \mathbf{x}_j|} \ . \tag{69}$$

But how to compute the energy stored in a continuous charge distribution? This question leads through deep waters to a suggestive answer. We plunge in with the Ansatz

$$W = \frac{1}{8\pi\epsilon_0} \int\int' \frac{\rho(\mathbf{x})\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \mathrm{d}^3x \mathrm{d}^3x' \ . \tag{70}$$

This is the analogue of our double sum for point charges, except that we did not restrict the integrals so as to exclude the interaction of a charge with itself. For the point charges these terms must be excluded if one is to avoid a meaningless answer. In the continuous case we just continue to swim. Using the solution (49) for the electric potential in the first step, we transform the expression to

$$\begin{aligned} W = \frac{1}{2}\int \rho\ \Phi \mathrm{d}^3x = -\frac{\epsilon_0}{2}\int \nabla^2\Phi\Phi \mathrm{d}^3x = \\ = \frac{\epsilon_0}{2}\int \nabla\Phi\cdot\nabla\Phi \mathrm{d}^3x = \frac{\epsilon_0}{2}\int \mathbf{E}\cdot\mathbf{E}\ \mathrm{d}^3x\ . \end{aligned} \tag{71}$$

In the middle of the calculation a partial integration was performed, and it was assumed that the surface terms vanishes provided it is evaluated on a very large sphere. Thus there was a faintly non-local flavour to the calculation.

The suggestion—which is substantiated in chapter 6—is that the electric field *in itself* carries energy. And it is in fact a *better* answer than is eq. (69), because it is never negative. Eq. (69) takes only the interaction energy into account, while our final expression includes self-energy contributions. Indeed, the electric field from two point charges is

$$\mathbf{E} = \frac{q_1}{4\pi\epsilon_0}\frac{\mathbf{x}-\mathbf{x}_1}{|\mathbf{x}-\mathbf{x}_1|^3} + \frac{q_2}{4\pi\epsilon_0}\frac{\mathbf{x}-\mathbf{x}_2}{|\mathbf{x}-\mathbf{x}_2|^3}\ . \tag{72}$$

Squaring that, eq. (71) gives the energy density as

$$w = \frac{1}{32\pi^2\epsilon_0}\left(\frac{q_1^2}{|\mathbf{x}-\mathbf{x}_1|^4} + \frac{q_2^2}{|\mathbf{x}-\mathbf{x}_2|^4} + \frac{2q_1q_2(\mathbf{x}-\mathbf{x}_1)\cdot(\mathbf{x}-\mathbf{x}_2)}{|\mathbf{x}-\mathbf{x}_1|^3|\mathbf{x}-\mathbf{x}_2|^3}\right)\ . \tag{73}$$

The last term is the interaction energy density. Integrating it over all space gives the total interaction energy

$$\begin{aligned} W_{\mathrm{int}} = \frac{q_1q_2}{16\pi^2\epsilon_0}\int \frac{(\mathbf{x}-\mathbf{x}_1)\cdot(\mathbf{x}-\mathbf{x}_2)}{|\mathbf{x}-\mathbf{x}_1|^3|\mathbf{x}-\mathbf{x}_2|^3}d^3x = \\ = \frac{q_1q_2}{16\pi^2\epsilon_0}\int \nabla\left(\frac{1}{|\mathbf{x}-\mathbf{x}_1|}\right)\cdot\nabla\left(\frac{1}{|\mathbf{x}-\mathbf{x}_2|}\right)d^3x = \\ = -\frac{q_1q_2}{16\pi^2\epsilon_0}\int \frac{1}{|\mathbf{x}-\mathbf{x}_1|}\nabla^2\left(\frac{1}{|\mathbf{x}-\mathbf{x}_2|}\right)d^3x = \\ = \frac{q_1q_2}{4\pi\epsilon_0}\int \frac{1}{|\mathbf{x}-\mathbf{x}_1|}\delta^{(3)}(\mathbf{x}-\mathbf{x}_2)d^3x = \frac{1}{4\pi\epsilon_0}\frac{q_1q_2}{|\mathbf{x}_1-\mathbf{x}_2|}\ . \end{aligned} \tag{74}$$

Here the partial integration led to a surface term that obviously vanishes. The result agrees with eq. (69). Notice that if we decide to include the self-energy terms in the energy density, the total energy will diverge. In fact you will encounter the integral

$$W \sim \int_0^\infty \frac{r^2}{r^4} \mathrm{d}r = \infty \ . \tag{75}$$

This simply shows that the notion of 'point charge' is problematic.

3.3 A VARIATIONAL PRINCIPLE

Let us have another look at the energy integral

$$W[\Phi] = \frac{\epsilon_0}{2} \int_V \mathbf{E} \cdot \mathbf{E} dV = \frac{\epsilon_0}{2} \int_V \nabla\Phi \cdot \nabla\Phi \mathrm{d}V \ . \tag{76}$$

In a wide variety of circumstances the static equilibrium state of some system can be found by minimizing its energy function. Suppose that the volume V is free of charges, and that the potential on its boundary S is fixed by Dirichlet boundary conditions. Then we can ask what choice of the function Φ inside V minimizes the energy integral? If you know how to do it, the question is easily answered. We are looking for that function Φ for which

$$I[\Phi + \delta\Phi] - I[\Phi] = 0 \ , \tag{77}$$

where the variation $\delta\Phi$ is a small but otherwise arbitrary function on V, except that we assume that $\delta\Phi = 0$ on the boundary because there the value of Φ is fixed by the boundary conditions. With this understanding it is easy to bring the variation to the form

$$I[\Phi + \delta\Phi] - I[\Phi] = \epsilon_0 \int_V \left(\nabla \cdot (\delta\Phi \nabla\Phi) - \delta\Phi \nabla^2 \Phi \right) \mathrm{d}V \ . \tag{78}$$

(We calculate to first order in $\delta\Phi$. If you are unfamiliar with this kind of calculation, just ignore it and go directly to the result.) Using Gauss' law, and then the fact that $\delta\Phi = 0$ on the boundary S, this is

$$I[\Phi+\delta\Phi]-I[\Phi] = \epsilon_0 \int_S \delta\Phi \nabla\Phi \cdot \mathbf{da} - \epsilon_0 \int \delta\Phi \nabla^2 \Phi dV = -\int \epsilon_0 \delta\Phi \nabla^2 \Phi \mathrm{d}V \ . \tag{79}$$

The energy integral has a minimum for those functions Φ such that the left hand side vanishes. The only way that this can happen for arbitrary choices of the variation $\delta\Phi$ inside the volume is that

$$\nabla\Phi = 0 \ . \tag{80}$$

In this way we see that Maxwell's electrostatic vacuum equation follows from minimization of the electrostatic energy.

In this calculation we assumed Dirichlet boundary conditions. We see immediately that if $\Phi = 0$ on the boundary then Φ has to vanish throughout the entire volume if $I[\phi]$ is to take its minimum value. Although we did not stress it, another way to guarantee uniqueness of the solution is to specify the normal derivative of the potential on the boundary. This is called Neumann boundary conditions. Jackson handles them by adding a surface term to the energy integral.

Now let us add a fixed charge density ρ inside the volume. At the expense of slightly confusing the connection to energy, we then look for the minimum of the integral

$$I[\Phi] = \frac{1}{2}\int_V (\epsilon_0 \nabla\Phi \cdot \nabla\Phi - 2\Phi\rho)\,\mathrm{d}V \ . \tag{81}$$

Assume Dirichlet boundary conditions and repeat the previous argument. We then find that the integral takes its minimum value for those functions Φ that obey Poisson's equation

$$\epsilon_0 \nabla^2\Phi + \rho = 0 \ . \tag{82}$$

This way of looking at things is very useful in situations where V or ρ are such that we are unable to solve Poisson's equation. We just choose *any* function Φ depending on a number of free parameters $\alpha_1, \alpha_2, \ldots, \alpha_n$. Then we compute the integral numerically and get an answer as a function of these parameters. Finally we minimize the resulting function $I = I(\alpha_1, \alpha_2, \ldots, \alpha_n)$ with respect to those parameters. The result will give our best approximation of the true Φ. With a computer we can afford a fairly large number of parameters, and some judgment in the choice of trial functions will often result in strikingly good approximations (as you can see in an example, and a Figure, given in Jackson's section 1.12).

3.4 GREEN FUNCTIONS

Green functions play an important role throughout Jackson's book, and a pedestrian look at what they are may be helpful. They are, in fact, inverses of linear partial differential operators. Suppose we are given the equation

$$\nabla^2\Phi(\mathbf{x}) = f(\mathbf{x}) \ , \tag{83}$$

where $f(\mathbf{x})$ is a known function. We want to find an inverse of the operator so that

$$\Phi(\mathbf{x}) = \text{“ } \frac{1}{\nabla^2} f(\mathbf{x}) \text{ ”} = \int' G(\mathbf{x}, \mathbf{x}') f(\mathbf{x}') \mathrm{d}^3 x' \ . \tag{84}$$

This requires that the Green function G obeys

$$\nabla^2 G(\mathbf{x}, \mathbf{x}') = \delta(\mathbf{x}, \mathbf{x}') \ . \tag{85}$$

The question is whether such an inverse exists, and if so if it is unique.

The operators are linear, and the functions on which the operators act are best thought of as vectors in some (infinite dimensional) function space. This means that we can get a feeling for things by recalling the conditions under which a matrix M has a unique inverse, that is when we can write

$$\sum_{j=1}^{n} M_{ij} v_j = f_i \qquad \Leftrightarrow \qquad v_i = \sum_{j=1}^{n} M^{-1}_{ij} f_j \ . \tag{86}$$

Instead of integrals we have finite sums, but the ideas are the same.

The inverse will fail to exist if the determinant is zero. Another way of saying this is that the inverse will fail to exist if there exist vectors $\mathbf{u}$ such that

$$\sum_{j=1}^{n} M_{ij} u_j = 0 \ . \tag{87}$$

What do we do if this happens? If the eigenvectors of M are orthogonal to each other we can restrict ourselves to a *subspace* of the original vector space. We can define a unique inverse matrix M^{-1} restricted to the subspace orthogonal to all zero eigenvectors. With this understanding eq. (86) is perfectly legitimate whenever $\mathbf{f}$ belongs to that subspace. Having defined M^{-1} we get solutions to $M\mathbf{v} = \mathbf{f}$ of the form

$$v_i = \sum_{j=1}^{n} M^{-1}_{ij} f_j + u_i \ . \tag{88}$$

The ambiguity in this solution disappears if we insist that $\mathbf{v}$ must lie in a subspace that contains no vectors $\mathbf{u}$ obeying $M\mathbf{u} = 0$.

For the Laplace operator we do have a problem with zero eigenvalues, because there do exist functions F such that

$$\nabla^2 F(\mathbf{x}) = 0 \ . \tag{89}$$

Therefore the Green function we have used so far is under suspicion. The most general Green function associated to the Laplace operator is of the form

$$G(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|} + F(\mathbf{x},\mathbf{x}') \ . \tag{90}$$

Provided that $F(\mathbf{x},\mathbf{x}')$ solves the Laplace equation this is a Green function obeying

$$\nabla^2 G(\mathbf{x},\mathbf{x}') = \nabla'^2 G(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x},\mathbf{x}') \ . \tag{91}$$

The freedom in choosing the function F reflects the freedom in choosing boundary conditions. Previously we could afford to set $F = 0$ because we interested in situations where the field vanishes at infinity, and we restricted ourselves to the space of regular functions falling off sufficiently fast at infinity. When you add two such functions you get a third function of this type, so this is indeed a linear subspace of the space of all functions.

In the general boundary value problem we may have to impose other restrictions on the space of allowed functions. Effectively we will use the homogeneous solution F to adapt the Green function to the problem at hand. If you find the argument below a little hard to follow, don't worry. You will understand it much better when we come to the method of images in chapter 2.

3.5 THE BOUNDARY VALUE PROBLEM REVISITED

We discussed conditions under which solutions of the Laplace equation are unique using the Mean Value Principle. This is not only elegant, it is practically useful too, as becomes clear in Jackson's section 1.13 where he presents a numerical method for how to solve the equation. But Jackson himself discusses uniqueness of the solutions along different lines, using Green functions.

First we bring in *Green's identity*. Because of Gauss' theorem there holds, for arbitrary functions f and g,that

$$\int_V (f\nabla^2 g - g\nabla^2 f)\mathrm{d}V = \int_V \nabla\cdot(f\nabla g - g\nabla f)\mathrm{d}V = \oint (f\nabla g - g\nabla f)\cdot\mathbf{da} \ . \tag{92}$$

Before making use of it we change the integration variable to $\mathbf{x}'$. Then we set $f(\mathbf{x}') = \Phi(\mathbf{x}')$ and $g(\mathbf{x}') = G(\mathbf{x},\mathbf{x}')$, where

$$\nabla'^2\Phi(\mathbf{x}') = -\frac{1}{\epsilon_0}\rho(\mathbf{x}') \ , \qquad \nabla'^2 G(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x},\mathbf{x}') \ . \tag{93}$$

The left hand side of Green's identity then evaluates to

$$-4\pi\Phi(\mathbf{x}) + \frac{1}{\epsilon_0}\int_V G(\mathbf{x},\mathbf{x}')\rho(\mathbf{x}')\mathrm{d}V' \ . \tag{94}$$

The right hand side becomes

$$\oint (\Phi(\mathbf{x}')\nabla' G(\mathbf{x},\mathbf{x}') - G(\mathbf{x},\mathbf{x}')\nabla'\Phi(\mathbf{x}')) \cdot \mathbf{da}' \ . \tag{95}$$

Now comes the trick. We assume that we can choose the so far arbitrary function F in the definition of the Green function so that the Green function vanishes on the boundary S of the region we consider. Then one of the terms in the surface integral vanishes. When we collect things together Green's identity gives us the equation

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int G(\mathbf{x},\mathbf{x}')\rho(\mathbf{x}')\mathrm{d}V' - \frac{1}{4\pi}\oint_S \Phi(\mathbf{x}')\nabla' G(\mathbf{x},\mathbf{x}') \cdot \mathbf{da}' \ . \tag{96}$$

Provided we can actually find the function F that we want, this is a fully explicit formula for the potential inside the volume in terms of the known charge density and the Dirichlet boundary conditions on the potential.

A variation is possible, where we choose the function F so that the normal derivative of the Green function vanishes on the boundary. Then we obtain

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int G(\mathbf{x},\mathbf{x}')\rho(\mathbf{x}')\mathrm{d}V' + \frac{1}{4\pi}\oint_S G(\mathbf{x},\mathbf{x}')\nabla'\Phi(\mathbf{x}') \cdot \mathbf{da}' \ . \tag{97}$$

This is a fully explicit formula where the potential is determined by the normal derivative of the potential on the boundary, that is to say by Neumann boundary conditions.

The bounding surface S may be disconnected—it may for instance consist of two concentric spheres, in which case the region lies in between. And in fact the outer sphere may be moved to infinity, in which case the condition that the potential vanishes at infinity serves as the appropriate Dirichlet condition at that end. We just have to insist that the boundary on which the conditions are set really does enclose the region where we want to solve the equation. We can impose Dirichlet conditions on one part of the boundary and Neumann conditions on another, but trying to impose both conditions at the same point will in general lead to inconsistencies. Evidently so, because once Dirichlet conditions (say) are imposed, the field including the Neumann conditions are completely determined so the latter can no longer be chosen freely.

In practice the explicit formulas in terms of the adapted Green function are difficult to use. If the boundary has an involved shape it will become very hard to solve for the function F, which is what we need to adapt the Green function. But for 'nice' boundaries we will see that the method of images can make life very easy.

Meanwhile we repeat the main mathematical message of this chapter:

Consider a closed surface S enclosing a region. Suppose $\nabla^2\Phi = 0$ throughout this region. Then Φ is uniquely determined by either Dirichlet or Neumann conditions on S.

On the whole we have avoided the question of existence of a solution. It is known that very general boundary conditions, on very general boundaries, do imply the existence of an analytic harmonic function in the interior. We just observe that a key idea in the proof is to focus on the existence of a minimum of the energy integral (76), and leave it at that.