83-10

March, 1983

Institute of Theoretical Physics
S-412 96 GÖTEBORG
Sweden

Cubic Interaction Terms for Arbitrary Spin

by

Anders K.H. Bengtsson
Ingemar Bengtsson
Lars Brink*

ABSTRACT

Using the light-cone formulation of dynamics, we construct cubic self interaction terms for massless fields of arbitrary helicity λ. The coupling constants have dimension $[\text{length}]^{\lambda-1}$ and there is a gauge group for any odd λ.

* Work supported in part by the Swedish Natural Science Research Council under contract No S-Fo 8115-109.

1. Introduction

There are two kinds of symmetry generators in physics, kinematical and dynamical ones. The latter, which transform the system forward in time, will in general be non-linearly (and non-locally) realized on fields since their canonical expressions are modified by interaction terms. To obtain linear representations of dynamical symmetries it will be necessary to add auxiliary fields to the action, as is done for example when a four-vector is used to describe a helicity-one field. Unfortunately, there is no principle guaranteeing the existence of these auxiliary fields. In fact, higher dimensional gravity theories are known, which seem to lack such a manifestly Lorentz covariant form[1]. This problem is also encountered in supersymmetry where the supersymmetry generators are dynamical within conventional spacelike quantization. For the particularly interesting case of the N=4 Yang-Mills theory quite strong no-go theorems [2] seem to preclude the existence of a reasonable set of auxiliary fields for the full set of supersymmetry generators. A manifestly supersymmetric form of this theory nevertheless exists [3], but it has not been possible to introduce interactions within this formulation.

In the case of the N=4 model this problem was recently bypassed [4] through the use of Dirac's [5] light-cone (null plane) formulation of dynamics. On the light cone a part of each supersymmetry generator becomes kinematical, so that no auxiliary fields are needed in order to obtain a superfield formulation of the theory. For a further discussion of the advantages of light-cone quantization within this context, see Ref. 6. In this paper we intend to use the fact that the light-cone formulation permits one to work with the physical fields exclusively (this is not always the case when the quantization surface is space-like) in order to discuss another theoretical problem exhibiting great similarity with the situation in the N=4 supersymmetric Yang-Mills theory. We will consider the question whether it is possible to have interactions in theories with helicity greater than 2. Again, manifestly covariant forms of such theories are known [7], but - apart from coupling to an external field - it has not been possible to introduce interactions within these formulations. For reviews of this question containing further references, see Ref. 8. In this paper we will argue - by exhibiting consistent three point couplings for all massless fields with integer helicity - that tnis is purely an artefact of the insistence on manifest covariance and not a property of higher spins as such.

The paper is organized as follows: In Sect. 2 we introduce our notation and define free fields of arbitrary helicity. In Sect. 3 we construct cubic self interaction terms for all theories where such terms are possible. Sec. 4 summarizes our conclusions as well as further problems. The appendix contains the Poincaré group in a suitable form. For further calculational details, consult Ref. 9 (the case treated there is N=1 supergravity).

2. The Free Fields in the Light-Cone Frame

The fields that we will treat will all be described in the light-cone frame, i.e. we use for the Minkowski space

$$
\begin{aligned}
x^{\pm} &= \frac{1}{\sqrt{2}}(x^0 \pm x^3) \\
x &= \frac{1}{\sqrt{2}}(x^1 + i x^2)
\end{aligned}
\qquad (2.1)
$$

and similarly for the space-time derivatives. (Note that transverse indices are complexified.) In this frame x^+ is taken as the evolution parameter with p^- as the corresponding Hamiltonian. Since any massless particle with spin has two degrees of freedom we will describe such particles of a definite spin λ by a complex field $\phi(x)$. Furthermore we will choose the representation such that $\phi(x)$ has definite helicity $-\lambda$ and $\bar{\phi}$ the helicity $+\lambda$. On a free field we can represent the orbital part of the Poincaré algebra by (see Appendix for further notations)

$$
\begin{aligned}
& p^- = -i\frac{\partial\bar{\partial}}{\partial^+}\,, \quad p^+ = -i\partial^+\,, \quad p = -i\partial\,, \quad \bar{p} = -i\bar{\partial} \\
& j = x\bar{\partial} - \bar{x}\partial \\
& j^+ = i(x\partial^+ - x^+\partial) \\
& \bar{j}^+ = i(\bar{x}\partial^+ - x^+\bar{\partial}) \\
& j^- = i\left(x\frac{\partial\bar{\partial}}{\partial^+} - x^-\partial\right) \\
& \bar{j}^- = i\left(\bar{x}\frac{\partial\bar{\partial}}{\partial^+} - x^-\bar{\partial}\right) \\
& j^{+-} = i\left(x^-\partial^+ - x^+\frac{\partial\bar{\partial}}{\partial^+}\right) .
\end{aligned}
\qquad (2.2)
$$

In this representation we have made the substitution $\partial^- = \frac{\partial\bar\partial}{\partial^+}$. This is the mass shell condition when it acts on a free field. However, since this condition is linear in ∂^-, the algebra still closes. It should, though, be noted that the representation of the generators is not hermitian (apart from the helicity operator) unless the mass shell condition is used. This is in contrast to the ordinary covariant formulation, and does correspond to a different way of going off-shell. This is not altogether new since for example the spin generators in the Dirac case is of this type.

To transform a field with helicity λ we also have to take the spin part of the Lorentz algebra into account [9]. The non-trivial ones are

$$\begin{aligned} \delta_j\, \varphi &= i\omega_j\,(\ell - \lambda)\,\varphi \\ \delta_{j^-}\, \varphi &= i\bar\omega_-\left(\ell^- - i\lambda \frac{\partial}{\partial^+}\right)\varphi \\ \delta_{\bar j^-}\, \varphi &= i\omega_-\left(\bar\ell^- + i\lambda \frac{\bar\partial}{\partial^+}\right)\varphi \end{aligned} \qquad (2.3)$$

To illustrate how these formulae are derived, we consider the Maxwell field A^μ. Impose first the light-cone gauge $A^+=0$ and work out the compensating gauge transformation necessary to stay in the gauge when Lorentz transformations are performed. Then we can read off the resulting transformations of the physical degrees of freedom if we solve for the unphysical mode A^-.

In this way one can derive the transformations for any field with helicity λ. However, it is simpler to start with δ_j, knowing that the spin part has eigenvalue λ and then close the algebra.

Note in this context that the equations (2.3) are quite close to the way massless particles are introduced as representations of the Lorentz group [10].

Since the free equations of motion are

$$\partial^-\varphi = \frac{\partial\bar\partial}{\partial^+}\varphi \quad , \qquad (2.4)$$

the corresponding Hamiltonian can be written as

$$\delta_H\,\varphi = \frac{\partial\bar\partial}{\partial^+}\varphi \quad . \qquad (2.5)$$

It is for this form that we will search for interaction terms, which are consistent with the Lorentz algebra.

3. Three Point Couplings

The Lagrangian that we will attempt to construct is of the form

$$\mathcal{L} = \tfrac{1}{2}\bar\varphi \,\Box\, \varphi + \alpha\left(\varphi\varphi\bar\varphi + \bar\varphi\bar\varphi\varphi\right) + \mathcal{O}(\alpha^2) \quad , \tag{3.1}$$

where α is a coupling constant and derivatives are still to be put into the interaction terms. The ensuing equation of motion for ϕ is

$$\delta_H \varphi = \frac{\partial\bar\partial}{\partial^+}\varphi + \alpha\varphi\varphi + \alpha\bar\varphi\varphi + \mathcal{O}(\alpha^2) \ . \tag{3.2}$$

(When we write an infinitesimal transformation we leave out the infinitesimal parameter).

There will also be nonlinear contributions to $\delta_j{}^{+-}$, $\delta_j{}^{-}$ and $\delta_{\bar j}{}^{-}$, as follows

$$\begin{aligned}
\delta_{j+-}\varphi &= \delta^{\circ}_{j+-}\varphi - i x^+ \delta^{\alpha}_H \varphi + \mathcal{O}(\alpha^2) \\
\delta_{j-}\varphi &= \delta^{\circ}_{j-}\varphi + i x\, \delta^{\alpha}_H \varphi + \delta^{\alpha}_s \varphi + \mathcal{O}(\alpha^2) \\
\delta_{\bar j -}\varphi &= \delta^{\circ}_{\bar j -}\varphi + i \bar x\, \delta^{\alpha}_H \varphi + \delta^{\alpha}_{\bar s}\varphi + \mathcal{O}(\alpha^2) \ .
\end{aligned} \tag{3.3}$$

With δ^α we mean the term to order α in the transformation.

Here, δ^α_s and $\delta^\alpha_{\bar s}$ are dynamical spin transformations of the form

$$\delta^{\alpha}_s \varphi = \alpha\varphi\varphi \quad , \qquad \delta^{\alpha}_{\bar s}\varphi = \alpha\bar\varphi\varphi \ . \tag{3.4}$$

The last equations are assumptions at this stage. They are, however, true when $\lambda=1,2$ and we will look for a similar solution in the general case.

Our task now is to find the Hamiltonian (3.2) together with the non-linear Lorentz transformations (3.3) such that they satisfy the correct algebra to first order in α. We then observe that terms of the type $\bar\phi\phi$ and $\phi\phi$ do not mix in the transformations to this order. We can hence reduce the amount of labour involved by first considering only terms of the latter kind. We will make the Ansatz that δ^α_H is a sum of terms of the form

$$\delta^{\alpha}_H \varphi = \alpha A \partial^{+\mu}\left[\bar\partial^{a}\partial^{+\rho}\varphi\, \bar\partial^{b}\partial^{+\sigma}\varphi\right] \quad , \tag{3.5}$$

where μ, ρ, σ, a and b are integers to be determined.

This is not the most general form, but again it agrees with the known solutions for $\lambda=1,2$. We must now investigate what restrictions the Lorentz algebra imposes on this form. We begin by checking the algebra of δ_H with the Lorentz generators and start with the kinematical transformation δ_j

$$\left[\delta_j, \delta_H\right]^{\alpha}\varphi = 0 \ . \tag{3.6}$$

Written out in detail, this becomes

$$\begin{aligned}
&\delta_j\, \alpha A \partial^{+\mu}\left[\bar\partial^{a}\partial^{+\rho}\varphi\,\bar\partial^{b}\partial^{+\sigma}\varphi\right] - \delta^{\alpha}_H(\ell-\lambda)\varphi = \\
&= \alpha A \partial^{+\mu}\left[\bar\partial^{a}\partial^{+\rho}(\ell-\lambda)\varphi\,\bar\partial^{b}\partial^{+\sigma}\varphi + \bar\partial^{a}\partial^{+\rho}\varphi\,\bar\partial^{b}\partial^{+\sigma}(\ell-\lambda)\varphi\right] + \\
&\quad -(\ell-\lambda)\alpha A\partial^{+\mu}\left[\bar\partial^{a}\partial^{+\rho}\varphi\,\bar\partial^{b}\partial^{+\sigma}\varphi\right] = \\
&= \alpha A\partial^{+\mu}\left[[\bar\partial^{a},\ell]\partial^{+\rho}\varphi\,\bar\partial^{b}\partial^{+\sigma}\varphi + \bar\partial^{a}\partial^{+\rho}\varphi\,[\bar\partial^{b},\ell]\partial^{+\sigma}\varphi\right] + \\
&\quad -\lambda\alpha A\partial^{+\mu}\left[\bar\partial^{a}\partial^{+\rho}\varphi\,\bar\partial^{b}\partial^{+\sigma}\varphi\right] = \\
&= (a+b-\lambda)\alpha A\partial^{+\mu}\left[\bar\partial^{a}\partial^{+\rho}\varphi\,\bar\partial^{b}\partial^{+\sigma}\varphi\right] \ .
\end{aligned} \tag{3.7}$$

Thus we get our first restriction on the derivative structure in (3.5) namely

$$a+b=\lambda\ . \tag{3.8}$$

We continue in this way. The commutator with $\delta_{j}{}^{+-}$ yields

$$\mu+\rho+\sigma=-1\ , \tag{3.9}$$

while the one with $\delta_{\bar{j}}{}^{+}$ is trivial. The commutator with $\delta_{j}{}^{+}$ on the other hand will mix terms in δ_H with different derivative structure, restricting the values of the coefficients A. In fact, with $\delta_H^\alpha\phi$ of the form (3.5) we get

$$[\delta_{j^+},\delta_H]^\alpha\varphi= \tag{3.10}$$

$$=\alpha A\left[ia\partial^{+\mu}\left[\partial^{+(\rho+1)}\bar{\partial}^{(a-1)}\varphi\,\bar{\partial}^{b}\partial^{+\sigma}\varphi\right]+ib\partial^{+\mu}\left[\bar{\partial}^{a}\partial^{+\rho}\varphi\,\bar{\partial}^{(b-1)}\partial^{+(\sigma+1)}\varphi\right]\right].$$

We are also - because of equation (3.4) - in a position to do

$$[\delta_{\bar{j}^-},\delta_H]^\alpha\varphi=0= \tag{3.11}$$

$$=i(\mu+1-\lambda)\frac{\bar{\partial}}{\partial^+}\alpha A\partial^{+\mu}\left[\bar{\partial}^{a}\partial^{+\rho}\varphi\,\bar{\partial}^{b}\partial^{+\sigma}\varphi\right]+$$

$$+\alpha A\left[i(\rho+\lambda)\partial^{+\mu}\left[\bar{\partial}^{(a+1)}\partial^{+(\rho-1)}\varphi\,\bar{\partial}^{b}\partial^{+\sigma}\varphi\right]+i(\sigma+\lambda)\partial^{+\mu}\left[\bar{\partial}^{a}\partial^{+\rho}\varphi\,\bar{\partial}^{(b+1)}\partial^{+(\sigma-1)}\varphi\right]\right].$$

Since both of these commutators have to be zero, they give us a set of consistency conditions which in fact will suffice to determine $\delta_H^\alpha\phi$ for us. Let us go through them helicity by helicity. For $\lambda=0$ we have $a=b=0$. A solution is clearly $\mu=-1$, $\rho=\sigma=0$ and

$$\delta_H^\alpha\varphi=\alpha\frac{1}{\partial^+}[\varphi\varphi]\ , \tag{3.12}$$

which, of course, is the coupling term in a ϕ^3 theory.

For $\lambda=1$ the number of $\bar{\partial}$s is one. In this case it is impossible to get zero out of both (3.10) and (3.11) unless we introduce an antisymmetric structure constant f^{abc}. With $\mu=0$, we have the solution

$$\delta_H^\alpha\varphi^a=\alpha f^{abc}\left[\frac{\bar{\partial}}{\partial^+}\varphi^b\varphi^c-\varphi^b\frac{\bar{\partial}}{\partial^+}\varphi^c\right]\ . \tag{3.13}$$

It is gratifying to see how the need for a gauge group emerges in this non covariant formalism, and it is not difficult to check that this is the 3-point coupling for a non-abelian Yang-Mills field theory.

For $\lambda=2$, we have a solution with $\mu=1$, namely

$$\delta_H^\alpha\varphi=\alpha\partial^+\left[\frac{\bar{\partial}^2}{\partial^{+2}}\varphi\varphi-2\frac{\bar{\partial}}{\partial^+}\varphi\frac{\bar{\partial}}{\partial^+}\varphi+\varphi\frac{\bar{\partial}^2}{\partial^{+2}}\varphi\right]\ . \tag{3.14}$$

We recognize this as the three point coupling in gravity [12]. Note that if λ is half integer, no three point coupling can exist because of (3.8). We have now exhausted all known cases. Nothing special happens, however, when we increase λ. For $\lambda=3$, we have a solution with $\mu=2$, again provided we introduce the structure constant f^{abc}:

$$\delta_H^\alpha\varphi^a=\alpha f^{abc}\partial^{+2}\left[\frac{\bar{\partial}^3}{\partial^{+3}}\varphi^b\varphi^c-3\frac{\bar{\partial}^2}{\partial^{+2}}\varphi^b\frac{\bar{\partial}}{\partial^+}\varphi^c+3\frac{\bar{\partial}}{\partial^+}\varphi^b\frac{\bar{\partial}^2}{\partial^{+2}}\varphi^c-\varphi^b\frac{\bar{\partial}^3}{\partial^{+3}}\varphi^c\right]. \tag{3.15}$$

We can go on, increasing λ step by step. However, we already see Pascal's triangle [11] emerge and can write down the general solution.

For λ even:

$$\delta_H^\alpha\varphi=\alpha\sum_{n=0}^{\lambda}(-1)^n\binom{\lambda}{n}\partial^{+(\lambda-1)}\left[\frac{\bar{\partial}^{(\lambda-n)}}{\partial^{+(\lambda-n)}}\varphi\,\frac{\bar{\partial}^n}{\partial^{+n}}\varphi\right]\ , \tag{3.16}$$

For λ odd:

$$\delta_H^{\alpha}\varphi^a = \alpha f^{abc}\sum_{n=0}^{\lambda}(-1)^n\binom{\lambda}{n}\partial^{+(\lambda-1)}\left[\frac{\bar{\partial}^{(\lambda-n)}}{\partial^{+(\lambda-n)}}\varphi^b\frac{\bar{\partial}^n}{\partial^{+n}}\varphi^c\right] \quad (3.17)$$

It is straightforward to obtain the actions from here (see below). There is still some work left though. Referring back to (3.3) and (3.4), we see that we need a further dynamical piece $\delta_s^{\alpha}\phi$ in order to close

$$[\delta_{j^-}, \delta_H]^{\alpha}\varphi = 0 \quad (3.18)$$

Demanding that (3.18) holds we find the following equation for $\delta_s^{\alpha}\phi$:

$$2i\lambda\frac{\partial}{\partial^+}\delta_H^{\alpha}\varphi - \delta_{-i\lambda\frac{\partial}{\partial^+}}\delta_H^{\alpha}\varphi + \frac{\partial\bar{\partial}}{\partial^+}\delta_s^{\alpha}\varphi - \delta_H^{o}\delta_s^{\alpha}\varphi = 0 \; . \quad (3.19)$$

The solution is

$$\delta_s^{\alpha}\varphi = -2i\alpha\lambda\sum_{n=0}^{\lambda-1}(-1)^n\binom{\lambda-1}{n}\partial^{+(\lambda-1)}\left[\frac{\bar{\partial}^{(\lambda-1-n)}}{\partial^{+(\lambda-n)}}\varphi\frac{\bar{\partial}^n}{\partial^{+n}}\varphi\right] . \quad (3.20)$$

This should be antisymmetrized for odd λ, of course. There remains to check that δ_{j^-} so obtained obeys the Lorentz algebra. This is straight forward and easy except for

$$[\delta_{j^-}, \delta_{\bar{j}^-}]^{\alpha}\varphi = 0 \quad , \quad (3.21)$$

which is straightforward and tedious, requiring the use of various properties of Pascal's triangle, such as

$$\binom{\lambda-1}{n} + \binom{\lambda-1}{n-1} = \binom{\lambda}{n} \quad . \quad (3.22)$$

We can now write down actions which yield our equations of motion:

$$S = \int d^4x\left\{\frac{1}{2}\bar{\varphi}\Box\varphi + \alpha\left[\bar{\varphi}\sum_{n=0}^{\lambda}(-1)^n\binom{\lambda}{n}\partial^{+\lambda}\left(\frac{\bar{\partial}^{(\lambda-n)}}{\partial^{+(\lambda-n)}}\varphi\frac{\bar{\partial}^n}{\partial^{+n}}\varphi\right) + c.c.\right]\right\} + O(\alpha^2) \, . \quad (3.23)$$

For odd λ there is a structure constant. If we make use of the equality

$$\sum_{n=0}^{\lambda}(-1)^n\binom{\lambda}{n}\frac{\partial^n}{\partial^{+(n+1)}}\left[\frac{\partial^{(\lambda-n)}}{\partial^{+(\lambda-n)}}\bar{\varphi}\,\partial^{+\lambda}\varphi\right] = \frac{1}{\partial^{+(\lambda+1)}}\sum_{n=0}^{\lambda}(-1)^n\binom{\lambda}{n}\left[\frac{\partial^{(\lambda-n)}}{\partial^{+(\lambda-n)}}\bar{\varphi}\,\partial^{+(2\lambda-n)}\partial^n\varphi\right] \; , \quad (3.24)$$

we find that the full form of $\delta_H^{\alpha}\phi$ including the $\bar{\phi}\phi$-terms that we have so far neglected is

$$\delta_H^{\alpha}\varphi = \alpha\sum_{n=0}^{\lambda}(-1)^n\binom{\lambda}{n}\left[\partial^{+(\lambda-1)}\left(\frac{\bar{\partial}^{(\lambda-n)}}{\partial^{+(\lambda-n)}}\varphi\frac{\bar{\partial}^n}{\partial^{+n}}\varphi\right) + 2\frac{1}{\partial^{+(\lambda+1)}}\left(\frac{\partial^{(\lambda-n)}}{\partial^{+(\lambda-n)}}\bar{\varphi}\,\partial^{+(2\lambda-n)}\partial^n\varphi\right)\right] \; . \quad (3.25)$$

Again, it can be checked that the algebra closes. This will also determine the missing piece in $\delta_{\bar{j}}\phi$ to be

$$\delta_{\bar{s}}^{\alpha}\varphi = -2i\lambda\alpha\sum_{n=0}^{\lambda-1}(-1)^n\binom{\lambda-1}{n}\frac{1}{\partial^{+(\lambda+1)}}\left[\frac{\partial^{(\lambda-1-n)}}{\partial^{+(\lambda-n)}}\bar{\varphi}\,\partial^{+(2\lambda-n)}\partial^n\varphi + \right. \quad (3.26)$$

$$\left. + 3\frac{\partial^{(\lambda-1-n)}}{\partial^{+(\lambda-1-n)}}\bar{\varphi}\,\partial^{+(2\lambda-n-1)}\partial^n\varphi\right] .$$

4. Conclusions and Comments

Let us summarize our results. We have found light cone actions describing massless fields of arbitrary spin λ, which lead to classical field theories in which the non linearly realized Lorentz algebra closes to first order in the coupling constant α. These actions are

$$S=\int d^4x\left\{\frac{1}{2}\bar{\varphi}\Box\varphi+\alpha\sum_{n=0}^{\lambda}(-1)^n\binom{\lambda}{n}\left[\bar{\varphi}\,\partial^{+\lambda}\left(\frac{\bar{\partial}^{(\lambda-n)}}{\partial^{+(\lambda-n)}}\varphi\,\frac{\bar{\partial}^n}{\partial^{+n}}\varphi\right)+c.c.\right]+\mathcal{O}(\alpha^2)\right\}, \quad (4.1)$$

for λ even and

$$S=\int d^4x\left\{\frac{1}{2}\bar{\varphi}^a\Box\varphi^a+\alpha f^{abc}\sum_{n=0}^{\lambda}(-1)^n\binom{\lambda}{n}\left[\bar{\varphi}^a\partial^{+\lambda}\left(\frac{\bar{\partial}^{(\lambda-n)}}{\partial^{+(\lambda-n)}}\varphi^b\frac{\bar{\partial}^n}{\partial^{+n}}\varphi^c\right)+c.c.+\mathcal{O}(\alpha^2)\right\} \quad (4.2)$$

for λ odd. There is a gaugegroup for any odd λ and the coupling constants have dimension $|\text{length}|^{\lambda-1}$.

Several problems now present themselves, the first one being the question of higher orders in α. It would presumably be necessary to know these actions to all orders in α in order to prove that the higher spin theories in fact are consistent. This may well be a difficult problem, but we foresee no problems of principle. The fact that the action (4.1) is known in closed form for $\lambda=2$ [12] is clearly encouraging.

Another obvious question is whether the higher spin theories are realized in Nature. It has been argued by Weinberg [13], who used S-matrix arguments, that no long-range forces can be generated by fields of higher spin than 2. They may still be there, however, although they have so far escaped observation. We have nothing whatsoever to say about this. Clearly, the dimensionful coupling constants would be difficult to handle in a quantum theory.

Our conclusion is that the higher spin theories are likely to exist, at least as classical field theories, although they may not have a manifestly covariant form.

Acknowledgements

One of us (IB) wishes to thank Jan Willem van Holten for a discussion about previous approaches to higher spin theory. We also wish to thank Olof Lindgren for proving a theorem about Pascal's triangle not given in Ref. 11, and Arne Kihlberg for a discussion about the Lorentz group.

Appendix:

We will express our Lorentz generators in terms of the customary ones as $J^+ = \frac{1}{\sqrt{2}}(J^{+1} + iJ^{+2})$, $\bar{J}^+ = \frac{1}{\sqrt{2}}(J^{+1} - ij^{+2})$, and similarly for J^-. Moreover $J = J^{12}$ and $H = P^-$. We also use $P = P^1 + iP^2$ and $\bar{P} = P^1 - iP^2$. The nonzero brackets of the Poincaré group are

$$
\begin{array}{lll}
[H, J^{+-}] = iH & [H, J^+] = iP & [H, \bar{J}^+] = i\bar{P} \\
[P^+, J^{+-}] = -iP^+ & [P^+, J^-] = iP & [P^+, \bar{J}^-] = i\bar{P} \\
[P, \bar{J}^-] = iH & [P, \bar{J}^+] = iP^+ & [P, J] = -P \\
[\bar{P}, J^-] = iH & [\bar{P}, J^+] = iP^+ & [\bar{P}, J] = \bar{P} \\
[J^-, J^{+-}] = iJ^- & [J^-, \bar{J}^+] = -iJ^{+-} - J & [J^-, J] = -J^- \\
[\bar{J}^-, J^{+-}] = i\bar{J}^- & [\bar{J}^-, J^+] = -iJ^{+-} + J & [\bar{J}^-, J] = \bar{J}^- \\
[J^{+-}, J^+] = iJ^+ & [J^{+-}, \bar{J}^+] = i\bar{J}^+ & \\
[J^+, J] = -J^+ & [\bar{J}^+, J] = \bar{J}^+ &
\end{array}
$$

References

1. N. Marcus and J.H. Schwarz, Phys. Lett. 115B (1982)111.

2. W. Siegel and M. Roček, Phys. Lett. 105B (1981) 279;
B.E.W. Nilsson, Göteborg preprint 81-6, February 1981.
V.O. Rivelles and J.G. Taylor, King's College 82-0483.

3. M. Sohnius, K.S. Stelle and P. West, in Unification of the Fundamental Particle Interactions, Eds. S. Ferrara, J. Ellis and P. van Nieuwenhuizen, Plenum Press 1980.

4. L. Brink, O. Lindgren and B.E.W. Nilsson, Nuclear Physics B212 (1983), 401;
S. Mandelstam, Nuclear Physics B 213 (1983), 149.

5. P.A.M. Dirac, Rev. Mod. Phys. 26 (1949) 392.

6. I. Bengtsson, Göteborg preprint 82-40, November 1982.

7. C.R. Hagen and L.P.S. Singh, Phys. Rev. D9 (1974), 898;
C. Fronsdal, Phys. Rev. D18 (1978) 3624;
J. Fang and C. Fronsdal, Phys. Rev. D18 (1978) 3630.

8. Supergravity, Eds. P. van Nieuwenhuizen and D.Z. Freedman, North-Holland Publishing Comp. 1979.

9. A.K.H. Bengtsson, Göteborg, preprint 83-9, April 1983.

10. E.P. Wigner, Ann. Math. 40 (1939) 139.

11. Yang Hui, 1261;
Chu-Shih-Chieh, 1303;
B. Pascal, 1664.

12. J. Scherk and J.H. Schwarz, Gen. Rel. and Grav. 6 (1975) 537.

M. Kaku, Nuclear Physics B91 (1975) 99.

13. S. Weinberg, Phys. Rev. B135 (1964) 1049.